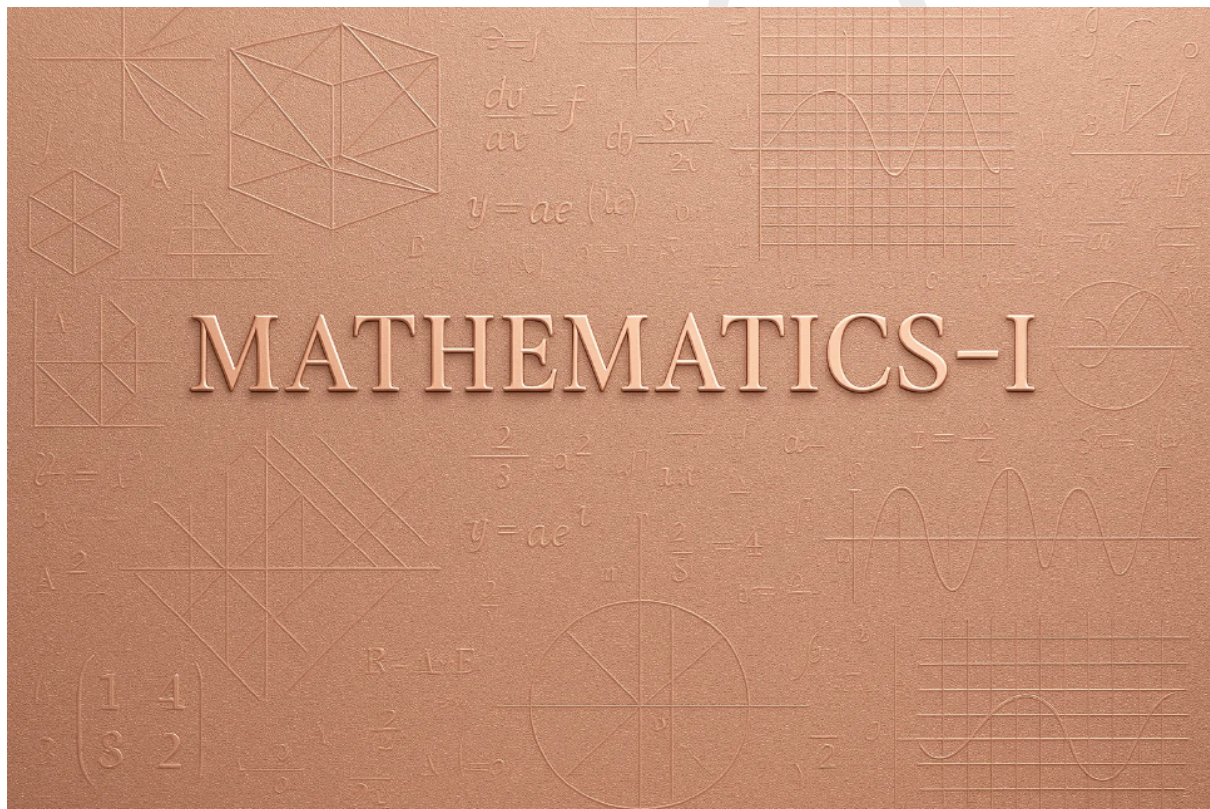


Mathematics-I (25SMT-121)

Compiled by : Subhayu





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Ch 1 : Basics of Calculus & Matrices - Introduction to matrices, Types of matrices, Orthogonal and Unitary matrices, Rank of matrices, Elementary transformations, reduction to echelon form

Ch 2 : Applications of Matrices - Consistency and solution of system of linear equations using matrices, Eigen values & Eigen Vectors, Linear dependence and independence of vectors, Cayley Hamilton Theorem (without proof), Inverse using Cayley Hamilton Theorem

Unit 2

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Ch 4 : Linear Transformations - Linear transformations (maps), range and kernel of a linear map, rank and nullity, Inverse of a linear transformation, rank- nullity theorem (Without Proof) , composition of linear maps, Matrix associated with a linear map

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Unit – 1

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Chapter 1

Basics of Calculus & Matrices

1.1 Introduction

Calculus and matrices are two fundamental pillars of modern mathematics. While calculus provides the language for understanding change and motion, matrices serve as the algebraic backbone for representing and solving systems of equations, transformations, and vector operations.

This chapter focuses on the foundational concepts of matrices — their types, properties, transformations, and methods for finding their rank and reduced forms — which play an essential role in higher mathematics, data analysis, and engineering computations.

1.2 Definition of a Matrix

A **matrix** is a rectangular arrangement of numbers (real or complex) in rows and columns. An $m \times n$ matrix has m rows and n columns, and is written as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Each a_{ij} represents the element in the i^{th} row and j^{th} column of matrix A .



1.3 Types of Matrices

1. Row and Column Matrices

A **row matrix** has only one row, e.g.,

$$[1 \ 2 \ 3]$$

A **column matrix** has only one column, e.g.,

$$\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

2. Square Matrix

A matrix with equal number of rows and columns ($m = n$) is called a **square matrix**.

Example:

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

3. Diagonal Matrix

A square matrix in which all non-diagonal elements are zero is called a **diagonal matrix**.

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

4. Scalar Matrix

If all diagonal elements of a diagonal matrix are equal, it is called a **scalar matrix**.

$$S = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

5. Identity Matrix

An **identity matrix** (or unit matrix) is a scalar matrix where $k = 1$. It is denoted by I_n :

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



6. Zero or Null Matrix

A matrix with all elements equal to zero.

$$O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

7. Symmetric and Skew-Symmetric Matrices

A matrix A is called:

- **Symmetric** if $A' = A$
- **Skew-Symmetric** if $A' = -A$

where A' denotes the transpose of A .

Example:

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \Rightarrow A' = A$$

Hence, A is symmetric.

$$B = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} \Rightarrow B' = -B$$

Hence, B is skew-symmetric.

1.4 Orthogonal and Unitary Matrices

1.4.1 Orthogonal Matrices

A square matrix A is said to be **orthogonal** if

$$A'A = AA' = I$$

which means that the transpose of A equals its inverse: $A' = A^{-1}$.

Example:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Here,

$$A' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad A'A = I$$

Thus, A is orthogonal.



1.4.2 Unitary Matrices

A complex square matrix A is said to be **unitary** if

$$A^*A = AA^* = I$$

where A^* is the conjugate transpose of A .

Example:

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

satisfies $U^*U = I$, hence U is unitary.

1.5 Rank of a Matrix

The **rank of a matrix** is the maximum number of linearly independent rows (or columns). It represents the dimension of the vector space spanned by its rows or columns.

1.5.1 Finding Rank by Echelon Form

A matrix is said to be in **echelon form** if:

- The first nonzero entry (called pivot) in each row occurs to the right of the pivot in the row above.
- Rows consisting entirely of zeros are at the bottom.

The number of nonzero rows in echelon form gives the rank.

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \Rightarrow \text{Apply } R_2 \rightarrow R_2 - 2R_1, \quad R_3 \rightarrow R_3 - R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -3 \\ 0 & -2 & 5 \end{bmatrix}$$

Now, apply $R_3 \rightarrow R_3 + 2R_2$:

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & -1 \end{bmatrix}$$

Hence, $\text{rank}(A) = 3$.



1.6 Elementary Transformations

Elementary transformations are operations used to simplify a matrix without changing its rank. They are:

1. Interchanging two rows (or columns)
2. Multiplying a row (or column) by a nonzero scalar
3. Adding a multiple of one row (or column) to another

These operations are fundamental in solving linear systems, finding inverses, and determining rank.

1.7 Reduction to Echelon Form

To reduce a matrix to echelon form:

1. Identify the leading entry (pivot) in each row.
2. Use elementary transformations to create zeros below each pivot.
3. Continue until no more eliminations are possible.

Example:

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 1 & 3 & 1 \\ 3 & 7 & 2 \end{bmatrix}$$

Applying:

$$R_2 \rightarrow R_2 - \frac{1}{2}R_1, \quad R_3 \rightarrow R_3 - \frac{3}{2}R_1$$

and then similar eliminations yield an upper triangular (echelon) matrix. The number of nonzero rows indicates the rank.

1.8 Applications of Matrices

Matrices play a crucial role in numerous domains:

- Solving systems of linear equations (via Gauss elimination or matrix inversion)
- Computer graphics (transformations and rotations)
- Network theory and Markov chains
- Machine learning (data representation and transformations)



1.9 Summary

- Matrices are rectangular arrays representing linear systems.
- Orthogonal matrices preserve length and angle under transformations.
- The rank of a matrix measures the number of independent rows/columns.
- Elementary transformations simplify matrices for analysis.
- Echelon form helps identify rank and solution consistency.

Boxed Formulae and Key Points

- $A' = A^{-1}$ for an orthogonal matrix.
- $A^* = A^{-1}$ for a unitary matrix.
- $\text{Rank}(A)$ = number of nonzero rows in echelon form.
- Symmetric matrix: $A' = A$; Skew-symmetric: $A' = -A$.
- Identity matrix I_n satisfies $AI_n = A$ for any $n \times n$ matrix.

Chapter 2

Applications of Matrices

2.1 Introduction

The study of matrices extends far beyond theoretical algebra — it provides a systematic method for solving linear systems, understanding geometric transformations, and analyzing complex networks. This chapter introduces some of the most powerful applications of matrices: solving systems of linear equations, computing eigenvalues and eigenvectors, examining vector dependence, and using the Cayley-Hamilton theorem to find matrix inverses efficiently.

2.2 Consistency and Solution of Linear Equations

A system of linear equations can be written in matrix form as:

$$AX = B$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Consistency of a System

A system $AX = B$ is said to be:

- **Consistent** if it has at least one solution.
- **Inconsistent** if it has no solution.



The consistency of the system is determined by comparing ranks:

$$\text{If Rank}(A) = \text{Rank}([A|B]) \Rightarrow \text{Consistent}$$

$$\text{If Rank}(A) \neq \text{Rank}([A|B]) \Rightarrow \text{Inconsistent}$$

Further:

- If $\text{Rank}(A) = \text{Rank}([A|B]) = n$, there is a **unique solution**.
- If $\text{Rank}(A) = \text{Rank}([A|B]) < n$, there are **infinitely many solutions**.

Example

Solve the system using matrix method:

$$\begin{cases} 2x + y + z = 10 \\ x + 3y + 2z = 13 \\ 3x + 2y + 4z = 20 \end{cases}$$

We write:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 3 & 2 & 4 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 10 \\ 13 \\ 20 \end{bmatrix}$$

$$AX = B$$

Compute A^{-1} and find $X = A^{-1}B$.

$$|A| = 2(12 - 4) - 1(4 - 6) + 1(2 - 9) = 16 - (-2) - 7 = 11$$

$$A^{-1} = \frac{1}{11} \begin{bmatrix} 8 & -2 & -1 \\ -2 & 5 & -2 \\ -1 & -2 & 5 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{11} \begin{bmatrix} 8 & -2 & -1 \\ -2 & 5 & -2 \\ -1 & -2 & 5 \end{bmatrix} \begin{bmatrix} 10 \\ 13 \\ 20 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Hence, the solution is $x = 2$, $y = 3$, $z = 4$.



2.3 Eigenvalues and Eigenvectors

For a square matrix A , a non-zero vector X is called an **eigenvector** if

$$AX = \lambda X$$

where λ is a scalar known as the **eigenvalue** of A .

Finding Eigenvalues

The eigenvalues are obtained by solving the characteristic equation:

$$|A - \lambda I| = 0$$

This is a polynomial equation in λ , known as the **characteristic equation**.

Example

Find eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 4 - \lambda & 2 \\ 1 & 3 - \lambda \end{vmatrix} = (4 - \lambda)(3 - \lambda) - 2 = \lambda^2 - 7\lambda + 10 = 0$$

$$\Rightarrow \lambda = 5, 2$$

For $\lambda = 5$:

$$(A - 5I)X = 0 \Rightarrow \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow x_2 = \frac{1}{2}x_1$$

$$\Rightarrow X_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

For $\lambda = 2$:

$$(A - 2I)X = 0 \Rightarrow \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow x_2 = -x_1$$

$$\Rightarrow X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Eigenvalues: 5, 2 Eigenvectors: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
--



2.4 Linear Dependence and Independence of Vectors

A set of vectors $\{X_1, X_2, \dots, X_n\}$ is said to be:

- **Linearly independent** if the equation

$$c_1X_1 + c_2X_2 + \dots + c_nX_n = 0$$

holds only when all constants $c_1 = c_2 = \dots = c_n = 0$.

- **Linearly dependent** if some constants are non-zero and the above relation holds.

Example:

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

Since $X_2 = 2X_1$, the vectors are linearly dependent.

2.5 Cayley-Hamilton Theorem

Statement: Every square matrix satisfies its own characteristic equation.

If the characteristic equation of A is:

$$|A - \lambda I| = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_n = 0$$

then according to the Cayley-Hamilton theorem:

$$A^n + a_1A^{n-1} + a_2A^{n-2} + \dots + a_nI = 0$$

Note: Proof is not included in this syllabus.

2.6 Finding Inverse Using Cayley-Hamilton Theorem

Steps:

1. Find the characteristic equation of A .
2. Replace λ by A in the equation.
3. Express A^{-1} in terms of A and I .



Example: Find A^{-1} using the Cayley-Hamilton theorem for:

$$A = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$$

Characteristic equation:

$$|A - \lambda I| = \begin{vmatrix} 4 - \lambda & 3 \\ 3 & 2 - \lambda \end{vmatrix} = \lambda^2 - 6\lambda - 1 = 0$$

Hence,

$$A^2 - 6A - I = 0 \Rightarrow A^2 - 6A = I$$

Multiplying both sides by A^{-1} :

$$A - 6I = A^{-1}$$

$$\boxed{A^{-1} = A - 6I}$$

2.7 Applications of Eigenvalues

- In differential equations, eigenvalues determine system stability.
- In data science, eigenvalues are central to Principal Component Analysis (PCA).
- In structural engineering, they define natural vibration frequencies.

Boxed Summary

- System $AX = B$ is consistent if $\text{Rank}(A) = \text{Rank}([A|B])$.
- Eigenvalues satisfy $|A - \lambda I| = 0$.
- Eigenvectors satisfy $(A - \lambda I)X = 0$.
- Cayley-Hamilton Theorem: A satisfies its own characteristic equation.
- A^{-1} can be found from Cayley-Hamilton by expressing I in terms of A .

Unit – 2

Ch 3 : Vector Space - Vector Space, Basis and dimension of a vector space

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Chapter 3

Vector Space

3.1 Introduction

In linear algebra, a **vector space** (also known as a **linear space**) is one of the most fundamental structures. It provides the foundation for various mathematical and computational concepts such as transformations, eigenvalues, and optimization.

A vector space is essentially a collection of objects called **vectors**, which can be added together and multiplied by scalars (numbers), while satisfying certain rules or axioms.

3.2 Definition of a Vector Space

Let F be a field (such as $\mathbb{R}$, the set of real numbers, or $\mathbb{C}$, the set of complex numbers). A **vector space** V over F is a non-empty set together with two operations:

- **Vector Addition:** An operation $+$ defined on V such that for all $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} + \mathbf{v} \in V$.
- **Scalar Multiplication:** An operation $\cdot$ defined such that for all $a \in F$ and $\mathbf{v} \in V$, $a \cdot \mathbf{v} \in V$.

These operations must satisfy the following **axioms of vector spaces** for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $a, b \in F$:

1. **Commutativity:** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. **Associativity:** $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. **Existence of Zero Vector:** There exists a $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$
4. **Existence of Additive Inverse:** For every $\mathbf{v} \in V$, there exists $-\mathbf{v} \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$



5. Distributivity of Scalar Multiplication:

$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$$

$$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$$

6. Compatibility: $(ab)\mathbf{v} = a(b\mathbf{v})$

7. Multiplicative Identity: $1\mathbf{v} = \mathbf{v}$, where 1 is the multiplicative identity in F

3.3 Examples of Vector Spaces

1. The set $\mathbb{R}^n$ (all n -tuples of real numbers) is a vector space over $\mathbb{R}$.
2. The set of all polynomials with real coefficients forms a vector space over $\mathbb{R}$.
3. The set of all $m \times n$ matrices with real entries forms a vector space under matrix addition and scalar multiplication.
4. The set of continuous functions $f(x)$ on the interval $[a, b]$ forms a vector space.

3.4 Subspace

A **subspace** W of a vector space V is a subset of V that is itself a vector space under the same operations of addition and scalar multiplication.

Conditions for Subspace: A non-empty subset W of V is a subspace if:

1. If $\mathbf{u}, \mathbf{v} \in W$, then $\mathbf{u} + \mathbf{v} \in W$
2. If $\mathbf{v} \in W$ and $a \in F$, then $a\mathbf{v} \in W$

Example

Let $V = \mathbb{R}^3$ and $W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$. Then W is a subspace of $\mathbb{R}^3$, since:

- It contains the zero vector $(0, 0, 0)$,
- It is closed under addition and scalar multiplication.



3.5 Linear Combination and Span

A **linear combination** of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ in V is any vector of the form:

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_n\mathbf{v}_n, \quad \text{where } a_i \in F$$

The set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is called their **span**, denoted as:

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$$

This span forms a subspace of V .

3.6 Linear Dependence and Independence

Definition: A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is said to be:

- **Linearly independent** if

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_n\mathbf{v}_n = \mathbf{0} \Rightarrow a_1 = a_2 = \dots = a_n = 0$$

- **Linearly dependent** if there exists a non-trivial combination of scalars such that:

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_n\mathbf{v}_n = \mathbf{0}$$

with at least one $a_i \neq 0$.

Example

The vectors $(1, 0)$ and $(0, 1)$ in $\mathbb{R}^2$ are linearly independent. However, $(1, 2)$ and $(2, 4)$ are linearly dependent because $(2, 4) = 2(1, 2)$.

3.7 Basis and Dimension

Basis

A **basis** of a vector space V is a set of linearly independent vectors that span V . That means every vector in V can be uniquely represented as a linear combination of the basis vectors.

If $V = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and the vectors are linearly independent, they form a basis.

**Example**

In $\mathbb{R}^2$, the set $\{(1, 0), (0, 1)\}$ is a basis because they are linearly independent and span the entire plane.

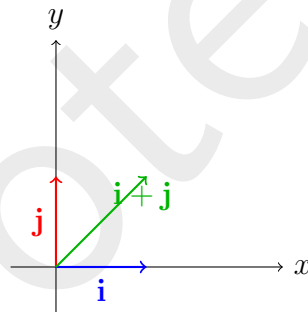
Dimension

The **dimension** of a vector space V , denoted as $\dim(V)$, is the number of vectors in any basis of V . All bases of a vector space have the same number of elements.

$$\dim(\mathbb{R}^n) = n$$

3.8 Geometric Interpretation

- In $\mathbb{R}^2$, the basis vectors define a plane through the origin.
- In $\mathbb{R}^3$, the basis vectors define a three-dimensional space.



Representation of a basis in $\mathbb{R}^2$

3.9 Key Takeaways

- A vector space consists of vectors that can be added and scaled.
- Subspaces are smaller vector spaces within a larger one.
- Linear independence ensures no vector can be formed from others.
- Basis defines the minimal set needed to span the space.
- The number of basis vectors gives the dimension.

Chapter 4

Linear Transformations

4.1 Introduction

Linear transformations are among the most important concepts in linear algebra. They describe how one vector space can be “mapped” or transformed into another while preserving the structure of addition and scalar multiplication.

In simple terms, a linear transformation takes vectors as input and produces new vectors as output — but in such a way that linear relationships between vectors remain unchanged.

4.2 Definition of a Linear Transformation

Let V and W be two vector spaces over the same field F .

A function (or map) $T : V \rightarrow W$ is called a **linear transformation** if, for all $\mathbf{u}, \mathbf{v} \in V$ and all scalars $a, b \in F$, the following two properties hold:

1. **Additivity:**

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

2. **Homogeneity (Scalar Multiplication):**

$$T(a\mathbf{u}) = aT(\mathbf{u})$$

If both properties are satisfied, T is said to be **linear**.

4.3 Examples of Linear Transformations

1. $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(x, y) = (x + y, 2y)$ is linear because both addition and scalar multiplication properties hold.

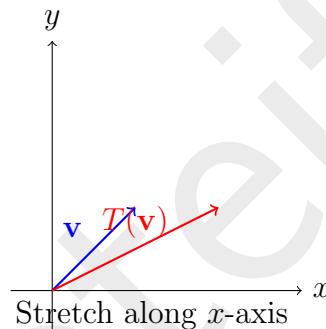


2. $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(x, y) = (x^2, y)$ is **not linear** since squaring the term violates homogeneity: $T(2x, 2y) = (4x^2, 2y) \neq 2T(x, y)$.

4.4 Geometric Interpretation

Linear transformations include common geometric operations such as:

- **Rotation** about the origin
- **Scaling** (stretching or shrinking)
- **Reflection** about an axis
- **Projection** onto a line or plane



Example: Linear transformation that stretches along the x -axis.

4.5 Range and Kernel of a Linear Transformation

Kernel (Null Space)

The **kernel** (or **null space**) of a linear transformation $T : V \rightarrow W$ is the set of all vectors in V that are mapped to the zero vector in W :

$$\ker(T) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\}$$

The kernel tells us which vectors are “annihilated” by T .

Example

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be defined by $T(x, y, z) = (x + y, y + z)$. Then $T(x, y, z) = (0, 0)$ gives:

$$x + y = 0, \quad y + z = 0$$

So, the kernel is $\ker(T) = \{(-y, y, -y) \mid y \in \mathbb{R}\}$.



Range (Image)

The **range** (or **image**) of T is the set of all vectors in W that are outputs of T :

$$\text{Range}(T) = \{T(\mathbf{v}) \mid \mathbf{v} \in V\}$$

The range represents all possible outcomes that T can produce.

Example

For the same transformation $T(x, y, z) = (x + y, y + z)$, the range is the entire $\mathbb{R}^2$ because any (a, b) can be achieved by choosing $x = a - y$ and $z = b - y$.

4.6 Rank and Nullity

The **rank** of T is the dimension of the range of T , and the **nullity** of T is the dimension of the kernel of T .

$$\text{rank}(T) = \dim(\text{Range}(T)), \quad \text{nullity}(T) = \dim(\ker(T))$$

Rank-Nullity Theorem (Without Proof)

For a linear transformation $T : V \rightarrow W$,

$$\text{rank}(T) + \text{nullity}(T) = \dim(V)$$

This theorem connects the structure of T with the dimensions of its domain and image.

4.7 Inverse of a Linear Transformation

A linear transformation $T : V \rightarrow W$ is said to be **invertible** if there exists another linear transformation $S : W \rightarrow V$ such that:

$$S(T(\mathbf{v})) = \mathbf{v}, \quad \forall \mathbf{v} \in V$$

and

$$T(S(\mathbf{w})) = \mathbf{w}, \quad \forall \mathbf{w} \in W$$

Then S is called the **inverse** of T and denoted as T^{-1} .



Example

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $T(x, y) = (2x, 3y)$. The inverse transformation is $T^{-1}(a, b) = \left(\frac{a}{2}, \frac{b}{3}\right)$.

4.8 Composition of Linear Maps

If $T_1 : V \rightarrow W$ and $T_2 : W \rightarrow U$ are linear transformations, then their **composition** is defined as:

$$(T_2 \circ T_1)(\mathbf{v}) = T_2(T_1(\mathbf{v}))$$

Properties:

- The composition of linear maps is also linear.
- Matrix representation of composition corresponds to matrix multiplication.

4.9 Matrix Associated with a Linear Map

Every linear transformation can be represented by a matrix, and this representation depends on the bases chosen for V and W .

If $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$, then there exists an $m \times n$ matrix A such that:

$$T(\mathbf{x}) = A\mathbf{x}$$

Example

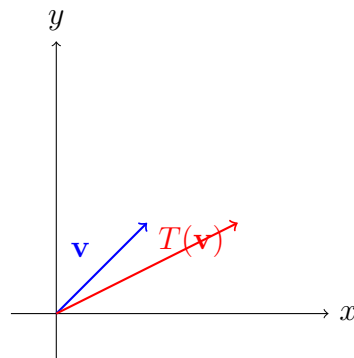
Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $T(x, y) = (3x + 2y, x - y)$. Then the matrix of T is:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$$

and $T(\mathbf{x}) = A\mathbf{x}$.



4.10 Geometric Visualization



Example: Linear map transforming $(1, 1)$ to $(2, 1)$ — a shearing transformation.

4.11 Summary

Key Takeaways

- A **linear transformation** preserves vector addition and scalar multiplication.
- The **kernel** represents all inputs mapped to zero.
- The **range** (or image) represents all possible outputs.
- **Rank** is the dimension of the image, **nullity** is the dimension of the kernel.
- The **Rank–Nullity Theorem** relates them: $\text{rank} + \text{nullity} = \dim(V)$.
- The matrix of a linear map defines the transformation algebraically.
- The composition of linear maps corresponds to matrix multiplication.
- An **inverse** exists only if the transformation is one-to-one and onto.

Unit – 3

Ch 5 : First Order Linear Differential Equations - Introduction to Differential equations: order and degree, Exact Differential equations, Equations reducible to exact form by integrating factors, Leibniz's linear differential equation and Bernoulli's differential equation

Ch 6 : Higher Order Linear Differential Equations - Methods of finding Complete Solutions for linear Differential Equation with constant coefficients: Complimentary functions, particular integrals: Method of variation of parameters to find the particular integrals, Linear Differential Equation with variable coefficient: Cauchy's homogeneous linear equation, Legendre's linear equation

Chapter 5

First Order Linear Differential Equations

5.1 Introduction to Differential Equations

A **differential equation** is an equation that involves one dependent variable, one or more independent variables, and the derivatives of the dependent variable with respect to the independent variables.

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots\right) = 0$$

5.1.1 Order and Degree

- **Order:** The order of a differential equation is the highest order derivative present in the equation.
- **Degree:** The degree is the power of the highest order derivative, after it is made free from radicals and fractions.

Example:

$$\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} = y$$

Order = 2, Degree = 3.

5.2 Exact Differential Equations

A first-order differential equation of the form

$$M(x, y) dx + N(x, y) dy = 0$$



is said to be **exact** if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

In that case, there exists a scalar function $\phi(x, y)$ such that:

$$\frac{\partial \phi}{\partial x} = M, \quad \frac{\partial \phi}{\partial y} = N$$

The general solution is:

$$\phi(x, y) = C$$

5.2.1 Solved Example 1:

$$(2xy + y^2) dx + (x^2 + 2xy) dy = 0$$

Here,

$$\frac{\partial M}{\partial y} = 2x + 2y, \quad \frac{\partial N}{\partial x} = 2x + 2y$$

Hence the equation is exact.

We integrate M partially with respect to x :

$$\phi(x, y) = \int M dx = \int (2xy + y^2) dx = x^2y + xy^2 + h(y)$$

Differentiating partially with respect to y :

$$\frac{\partial \phi}{\partial y} = x^2 + 2xy + h'(y)$$

But $\frac{\partial \phi}{\partial y} = N = x^2 + 2xy$, so $h'(y) = 0 \Rightarrow h(y) = C$.

Solution:

$$x^2y + xy^2 = C$$

5.3 Equations Reducible to Exact Form (Integrating Factor Method)

When $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is not exact. In such cases, it can often be made exact by multiplying both sides by a suitable function called the **Integrating Factor (I.F.)**.

$$\mu(x, y) (M dx + N dy) = 0$$



5.3.1 Common Rules to Find Integrating Factor:

- If $\frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)$ is a function of x only, then

$$\text{I.F.} = e^{\int P(x) dx}$$

- If $\frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$ is a function of y only, then

$$\text{I.F.} = e^{\int Q(y) dy}$$

5.3.2 Solved Example 2:

$$(2y - x) dx + (x + y) dy = 0$$

Here,

$$\frac{\partial M}{\partial y} = 2, \quad \frac{\partial N}{\partial x} = 1$$

Not exact since they are not equal.

$$\frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{1}{x+y} (1) = \frac{1}{x+y}$$

Hence,

$$\text{I.F.} = e^{\int \frac{dx}{x+y}} = e^{\ln|x+y|} = (x+y)$$

Multiplying the equation by $(x+y)$ makes it exact, after which it can be solved by the standard method for exact equations.

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5.4 Leibniz's Linear Differential Equation

A first order linear differential equation is of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Solution: The integrating factor is given by:

$$\text{I.F.} = e^{\int P(x) dx}$$

The general solution is:

$$y \times \text{I.F.} = \int Q(x) \times \text{I.F.} dx + C$$



5.4.1 Solved Example 3:

$$\frac{dy}{dx} + 3y = 6x$$

Here, $P = 3$, $Q = 6x$.

$$I.F. = e^{\int 3dx} = e^{3x}$$

Multiplying through by $I.F.$:

$$e^{3x} \frac{dy}{dx} + 3ye^{3x} = 6xe^{3x}$$

$$\Rightarrow \frac{d}{dx}(ye^{3x}) = 6xe^{3x}$$

Integrate both sides:

$$ye^{3x} = 6 \int xe^{3x} dx + C$$

Using integration by parts:

$$\int xe^{3x} dx = \frac{xe^{3x}}{3} - \frac{e^{3x}}{9}$$

$$\Rightarrow ye^{3x} = 6 \left(\frac{xe^{3x}}{3} - \frac{e^{3x}}{9} \right) + C$$

$$\Rightarrow y = 2x - \frac{2}{3} + Ce^{-3x}$$

5.5 Bernoulli's Differential Equation

A Bernoulli's equation is of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

It is nonlinear, but can be reduced to linear form by a substitution.

To Solve:

1. Divide through by y^n :

$$y^{-n} \frac{dy}{dx} + P(x)y^{1-n} = Q(x)$$

2. Substitute $z = y^{1-n}$ so that:

$$\frac{dz}{dx} = (1-n)y^{-n} \frac{dy}{dx}$$



3. Substitute and simplify to a linear equation in z :

$$\frac{dz}{dx} + (1 - n)P(x)z = (1 - n)Q(x)$$

4. Solve for z using integrating factor method, then back-substitute y .

5.5.1 Solved Example 4:

$$\frac{dy}{dx} + y = y^2 e^x$$

Here, $P = 1$, $Q = e^x$, $n = 2$.

Divide by y^2 :

$$y^{-2} \frac{dy}{dx} + y^{-1} = e^x$$

Let $z = y^{-1} \Rightarrow \frac{dz}{dx} = -y^{-2} \frac{dy}{dx}$

$$\Rightarrow -\frac{dz}{dx} + z = e^x$$

$$\Rightarrow \frac{dz}{dx} - z = -e^x$$

Integrating factor:

$$I.F. = e^{-\int 1 dx} = e^{-x}$$

$$ze^{-x} = -\int e^{-x} e^x dx + C = -\int 1 dx + C = -x + C$$

$$\Rightarrow z = e^x(-x + C)$$

$$\Rightarrow y = \frac{1}{z} = \frac{e^{-x}}{C - x}$$

5.6 Summary of Standard Forms

- **Linear:** $\frac{dy}{dx} + Py = Q \Rightarrow yI.F. = \int QI.F. dx + C$
- **Bernoulli:** $\frac{dy}{dx} + Py = Qy^n$
- **Exact:** $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
- **Reducible to Exact:** Multiply by suitable I.F.



5.7 Practice Numericals (Exam Oriented)

1. Solve $\frac{dy}{dx} + 2y = 3e^{-x}$.
2. Solve $(y - 2x) dx + (x + y) dy = 0$.
3. Solve $\frac{dy}{dx} + y \tan x = \sin x$, $0 < x < \frac{\pi}{2}$.
4. Solve $\frac{dy}{dx} + y = e^{2x}y^2$.
5. Find the general solution of $\frac{dy}{dx} = \frac{x + y}{x - y}$.

For all sums, assume constants as needed and show each integration step clearly. Marks are awarded for the correct form of *I.F.* and substitution.

Solutions to Practice Numericals

1. **Solve** $\frac{dy}{dx} + 2y = 3e^{-x}$.

This is a first-order linear ODE of the form $y' + P(x)y = Q(x)$ with $P(x) = 2$, $Q(x) = 3e^{-x}$.

$$\text{Integrating factor (I.F.)} = e^{\int 2 dx} = e^{2x}.$$

$$\text{Multiply through: } e^{2x}y' + 2e^{2x}y = 3e^{2x}e^{-x} = 3e^x.$$

$$\text{Left side is } \frac{d}{dx}(e^{2x}y) = 3e^x.$$

$$\text{Integrate: } e^{2x}y = 3 \int e^x dx + C = 3e^x + C.$$

$$\Rightarrow y = 3e^{-x} + Ce^{-2x}, \quad C \in \mathbb{R}.$$

2. **Solve** $(y - 2x) dx + (x + y) dy = 0$.

Write $M(x, y) = y - 2x$, $N(x, y) = x + y$. Check exactness:

$$M_y = 1, \quad N_x = 1 \quad \Rightarrow \quad M_y = N_x,$$

so the equation is exact.



We seek $\phi(x, y)$ with $\phi_x = M$ and $\phi_y = N$.

Integrate $\phi_x = y - 2x$ w.r.t. x :

$$\phi(x, y) = \int (y - 2x) dx = xy - x^2 + h(y).$$

Differentiate this w.r.t. y :

$$\phi_y = x + h'(y) \stackrel{!}{=} N = x + y \Rightarrow h'(y) = y.$$

Hence $h(y) = \frac{1}{2}y^2 + C_0$.

Thus

$$\phi(x, y) = xy - x^2 + \frac{1}{2}y^2 = C,$$

is the implicit general solution.

3. **Solve** $\frac{dy}{dx} + y \tan x = \sin x, \quad 0 < x < \frac{\pi}{2}.$

Linear ODE with $P(x) = \tan x$, $Q(x) = \sin x$.

$$\text{I.F.} = e^{\int \tan x dx} = e^{-\ln |\cos x|} = \sec x \quad (\text{on } (0, \frac{\pi}{2}) \cos x > 0).$$

$$\text{Multiply: } \sec x y' + \sec x \tan x y = \sin x \sec x = \tan x.$$

$$\text{Left side is } \frac{d}{dx}(y \sec x) = \tan x.$$

$$\text{Integrate: } y \sec x = \int \tan x dx = -\ln |\cos x| + C.$$

$$\Rightarrow y = \cos x (-\ln |\cos x| + C).$$

(On the given interval $0 < x < \frac{\pi}{2}$ we can drop the absolute value: $y = -\cos x \ln(\cos x) + C \cos x$.)

4. **Solve** $\frac{dy}{dx} + y = e^{2x}y^2.$

This is a Bernoulli equation with $n = 2$. Rewrite as

$$y' + y = e^{2x}y^2.$$

Divide both sides by y^2 (assuming $y \neq 0$):

$$y^{-2}y' + y^{-1} = e^{2x}.$$



Set $z = y^{-1} \Rightarrow \frac{dz}{dx} = -y^{-2}y'$. Therefore $y^{-2}y' = -z'$ and the equation becomes

$$-z' + z = e^{2x} \quad \Rightarrow \quad z' - z = -e^{2x}.$$

This is linear in z with integrating factor $I.F. = e^{\int -1 dx} = e^{-x}$.

$$\frac{d}{dx}(ze^{-x}) = -e^{2x}e^{-x} = -e^x.$$

$$\begin{aligned} \text{Integrate: } ze^{-x} &= -\int e^x dx + C = -e^x + C. \\ \Rightarrow z &= -e^{2x} + Ce^x. \end{aligned}$$

Recall $z = 1/y$, hence

$$y = \frac{1}{-e^{2x} + Ce^x} = \frac{e^{-x}}{C - e^x}, \quad C \in \mathbb{R}.$$

(Any equivalent algebraic rearrangement is acceptable.)

5. Find the general solution of $\frac{dy}{dx} = \frac{x+y}{x-y}$.

This is homogeneous of degree zero in (x, y) ; set $y = vx$ so $y' = v + xv'$.

$$v + x \frac{dv}{dx} = \frac{x + vx}{x - vx} = \frac{1 + v}{1 - v}.$$

Thus

$$x \frac{dv}{dx} = \frac{1 + v}{1 - v} - v = \frac{1 + v - v(1 - v)}{1 - v} = \frac{1 + v^2}{1 - v}.$$

Separate variables:

$$\frac{1 - v}{1 + v^2} dv = \frac{dx}{x}.$$

Integrate both sides. For the left side split into two integrals:

$$\int \frac{1 - v}{1 + v^2} dv = \int \frac{1}{1 + v^2} dv - \int \frac{v}{1 + v^2} dv = \arctan v - \frac{1}{2} \ln(1 + v^2).$$

So

$$\arctan v - \frac{1}{2} \ln(1 + v^2) = \ln|x| + C.$$

Substitute back $v = \frac{y}{x}$ to obtain an implicit solution:

$$\boxed{\arctan\left(\frac{y}{x}\right) - \frac{1}{2} \ln\left(1 + \left(\frac{y}{x}\right)^2\right) = \ln|x| + C.}$$



This can be simplified (combine the $\ln$ terms) to the equivalent form

$$\arctan\left(\frac{y}{x}\right) - \frac{1}{2} \ln(x^2 + y^2) = C',$$

which is often a more compact implicit general solution (where C' absorbs constants).

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Chapter 6

Higher Order Linear Differential Equations

6.1 Introduction

A **higher order linear differential equation** is one in which the dependent variable and its derivatives appear in a linear form, and the highest derivative is of order greater than one.

The general form of an n^{th} order linear differential equation with constant coefficients is:

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = f(x)$$

where:

- $a_0, a_1, \dots, a_n$ are constants,
- $f(x)$ is a given function of x .

If $f(x) = 0$, the equation is called a **homogeneous linear differential equation**. If $f(x) \neq 0$, it is called a **non-homogeneous linear differential equation**.

6.2 Standard Operator Form

Using the differential operator $D = \frac{d}{dx}$, the equation can be written as:

$$f(D)y = F(x)$$

where

$$f(D) = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0$$



and $F(x)$ corresponds to the non-homogeneous part.

Thus,

$$f(D)y = F(x)$$

The **complete solution (C.S.)** of the differential equation is given by:

$$\text{C.S.} = \text{C.F.} + \text{P.I.}$$

where:

- C.F. = Complementary Function (solution of $f(D)y = 0$)
- P.I. = Particular Integral (specific solution of $f(D)y = F(x)$)

6.3 Complementary Function (C.F.)

To find the C.F., solve the **auxiliary equation**:

$$f(m) = 0$$

where m replaces D .

6.3.1 Cases:

- **Case 1:** Distinct real roots $m_1, m_2, \dots, m_n$

$$\text{C.F.} = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \dots + C_n e^{m_n x}$$

- **Case 2:** Repeated roots If a root m repeats r times:

$$(D - m)^r y = 0 \Rightarrow \text{C.F.} = (C_1 + C_2 x + C_3 x^2 + \dots + C_r x^{r-1}) e^{mx}$$

- **Case 3:** Complex roots $a \pm ib$

$$\text{C.F.} = e^{ax} (C_1 \cos bx + C_2 \sin bx)$$



6.4 Particular Integral (P.I.)

The P.I. is given by:

$$\text{P.I.} = \frac{1}{f(D)} F(x)$$

6.4.1 Standard Rules for Finding P.I.

- **Rule 1:** If $F(x) = e^{ax}$, then

$$\text{P.I.} = \frac{e^{ax}}{f(a)}, \quad \text{provided } f(a) \neq 0$$

If $f(a) = 0$, multiply by x until the denominator becomes non-zero.

- **Rule 2:** If $F(x) = \sin(ax)$ or $\cos(ax)$, then

$$\text{P.I.} = \frac{\sin(ax)}{f(-a^2)} \text{ or } \frac{\cos(ax)}{f(-a^2)}$$

If $f(-a^2) = 0$, multiply by x until it is non-zero.

- **Rule 3:** If $F(x)$ is a polynomial in x , say x^n , Expand $\frac{1}{f(D)}$ by the Binomial Theorem and apply it term by term.

6.5 Method of Variation of Parameters

This is a powerful method for finding the **particular integral** when $F(x)$ is not of standard form.

For a second-order equation:

$$y'' + P(x)y' + Q(x)y = R(x)$$

If y_1 and y_2 are linearly independent solutions of the homogeneous equation, then the particular integral is given by:

$$y_p = -y_1 \int \frac{y_2 R}{W} dx + y_2 \int \frac{y_1 R}{W} dx$$

where W is the **Wronskian**:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$



6.5.1 Example

Solve:

$$y'' + y = \sec x$$

Solution: Homogeneous part: $y'' + y = 0 \Rightarrow$ roots $\pm i$ Hence, $y_1 = \cos x$, $y_2 = \sin x$, $W = 1$.

Then:

$$y_p = -\cos x \int \sin x \sec x dx + \sin x \int \cos x \sec x dx$$

$$y_p = -\cos x \int \tan x dx + \sin x \int dx = -\cos x \ln |\sec x| + \sin x$$

Hence,

$$y = C_1 \cos x + C_2 \sin x - \cos x \ln |\sec x| + \sin x$$

6.6 Linear Differential Equations with Variable Coefficients

6.6.1 Cauchy's Homogeneous Linear Equation

The general form is:

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n y = 0$$

To reduce it to constant coefficients, substitute:

$$x = e^t \Rightarrow \frac{d}{dx} = \frac{1}{x} \frac{d}{dt}$$

Hence,

$$x^n \frac{d^n y}{dx^n} = (D)(D-1)(D-2) \dots (D-n+1)y$$

where $D = \frac{d}{dt}$.

Example:

$$x^2 y'' - 3xy' + 4y = 0$$

Put $x = e^t \Rightarrow D = \frac{d}{dt}$

Then:

$$(D)(D-1)y - 3Dy + 4y = 0 \Rightarrow (D^2 - 4D + 4)y = 0$$

Auxiliary equation: $(m-2)^2 = 0 \Rightarrow m = 2, 2$

$$y = (C_1 + C_2 t)e^{2t} \Rightarrow y = (C_1 + C_2 \ln x)x^2$$



6.6.2 Legendre's Linear Differential Equation

It is of the form:

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$$

It commonly arises in physics (e.g., spherical harmonics).

For small n :

$$n = 0 : y = 1$$

$$n = 1 : y = x$$

$$n = 2 : y = \frac{1}{2}(3x^2 - 1)$$

These are called **Legendre polynomials** $P_n(x)$.

6.7 Summary Table

Concept	Formula / Method
General form	$f(D)y = F(x)$
Complementary Function	From auxiliary equation $f(m) = 0$
Particular Integral	$\frac{1}{f(D)}F(x)$
Cauchy's Equation	$x = e^t \Rightarrow D = \frac{d}{dt}$
Legendre's Equation	$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$
Variation of Parameters	$y_p = -y_1 \int \frac{y_2 R}{W} dx + y_2 \int \frac{y_1 R}{W} dx$

6.8 Practice Examples

1. Solve: $(D^2 - 3D + 2)y = e^{2x}$
2. Solve: $(D^2 + 4)y = \cos 2x$
3. Solve by variation of parameters: $y'' + y = \sec x$
4. Solve: $x^2 y'' - 4xy' + 6y = 0$
5. Solve: $(1 - x^2)y'' - 2xy' + 2y = 0$



Solutions of Practice Examples

6.8.1 1. Solve $(D^2 - 3D + 2)y = e^{2x}$.

The homogeneous (complementary) equation is

$$(D^2 - 3D + 2)y = 0$$

with auxiliary equation $m^2 - 3m + 2 = 0$. Thus $m = 1, 2$ and

$$y_{\text{C.F.}} = C_1e^x + C_2e^{2x}.$$

RHS is e^{2x} . Evaluate $f(a)$ where $f(m) = m^2 - 3m + 2$ and $a = 2$:

$$f(2) = 4 - 6 + 2 = 0,$$

so resonance occurs. Hence try a particular integral of the form

$$y_p = Axe^{2x}.$$

Compute derivatives:

$$y'_p = Ae^{2x} + 2Axe^{2x}, \quad y''_p = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x} = 4Ae^{2x} + 4Axe^{2x}.$$

Now substitute into the operator:

$$(D^2 - 3D + 2)y_p = (4Ae^{2x} + 4Axe^{2x}) - 3(Ae^{2x} + 2Axe^{2x}) + 2(Axe^{2x}).$$

Simplify:

$$= (4A - 3A)e^{2x} + (4A - 6A + 2A)xe^{2x} = Ae^{2x}.$$

Equate to RHS e^{2x} gives $A = 1$. Thus

$$y_p = xe^{2x}.$$

Therefore the complete solution is

$$y = C_1e^x + C_2e^{2x} + xe^{2x}.$$



6.8.2 2. Solve $(D^2 + 4)y = \cos 2x$.

Homogeneous equation:

$$(D^2 + 4)y = 0 \Rightarrow m^2 + 4 = 0 \Rightarrow m = \pm 2i,$$

hence

$$y_{\text{C.F.}} = C_1 \cos 2x + C_2 \sin 2x.$$

RHS is $\cos 2x$. Since $\cos 2x$ (and $\sin 2x$) are solutions of the homogeneous equation, resonance occurs. Try

$$y_p = x(a \cos 2x + b \sin 2x).$$

Differentiate:

$$\begin{aligned} y_p' &= a \cos 2x + b \sin 2x + x(-2a \sin 2x + 2b \cos 2x), \\ y_p'' &= -2a \sin 2x + 2b \cos 2x + (-2a \sin 2x + 2b \cos 2x) + x(-4a \cos 2x - 4b \sin 2x) \\ &= -4a \sin 2x + 4b \cos 2x + x(-4a \cos 2x - 4b \sin 2x). \end{aligned}$$

Now

$$(D^2 + 4)y_p = y_p'' + 4y_p = [-4a \sin 2x + 4b \cos 2x] + x[(-4a \cos 2x - 4b \sin 2x) + 4(a \cos 2x + b \sin 2x)].$$

The bracket multiplying x vanishes, so

$$(D^2 + 4)y_p = -4a \sin 2x + 4b \cos 2x.$$

Set equal to $\cos 2x$. Equate coefficients:

$$4b = 1 \Rightarrow b = \frac{1}{4}, \quad -4a = 0 \Rightarrow a = 0.$$

Thus

$$y_p = \frac{x}{4} \sin 2x,$$

and the general solution is

$$y = C_1 \cos 2x + C_2 \sin 2x + \frac{x}{4} \sin 2x.$$



6.8.3 3. Solve by variation of parameters: $y'' + y = \sec x$.

Homogeneous equation $y'' + y = 0$ has solutions

$$y_1 = \cos x, \quad y_2 = \sin x,$$

and the Wronskian $W = \det \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} = 1$.

Variation of parameters formula for a particular integral:

$$y_p = -y_1 \int \frac{y_2 R(x)}{W} dx + y_2 \int \frac{y_1 R(x)}{W} dx,$$

with $R(x) = \sec x$. Compute the integrals:

$$\int y_2 R dx = \int \sin x \sec x dx = \int \tan x dx = -\ln |\cos x|,$$

$$\int y_1 R dx = \int \cos x \sec x dx = \int 1 dx = x.$$

Therefore

$$y_p = -\cos x (-\ln |\cos x|) + \sin x \cdot x = \cos x \ln |\cos x| + x \sin x.$$

Thus the complete solution is

$$y = C_1 \cos x + C_2 \sin x + x \sin x + \cos x \ln |\cos x|.$$

(One may rewrite $\cos x \ln |\cos x| = -\cos x \ln |\sec x|$ if preferred.)

6.8.4 4. Solve the Cauchy–Euler equation $x^2 y'' - 4xy' + 6y = 0$.

For a Cauchy–Euler equation assume $y = x^m$. Substitute:

$$x^2 m(m-1)x^{m-2} - 4mx^{m-1} + 6x^m = [m(m-1) - 4m + 6]x^m = 0.$$

So the indicial (characteristic) equation is

$$m^2 - 5m + 6 = 0 \quad \Rightarrow \quad (m-2)(m-3) = 0,$$

hence $m = 2, 3$. Therefore

$$y = C_1 x^2 + C_2 x^3.$$



6.8.5 5. Solve $(1 - x^2)y'' - 2xy' + 2y = 0$ (Legendre type).

This is a Legendre differential equation of the form

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0,$$

with $n(n+1) = 2$. Solve $n^2 + n - 2 = 0$, giving $n = 1$ or $n = -2$. The physically standard (polynomial) solution corresponds to $n = 1$.

For $n = 1$ the *Legendre polynomial* is

$$P_1(x) = x,$$

so one solution is $y_1(x) = x$.

The general solution of Legendre's equation for integer n is a linear combination of the Legendre polynomial $P_n(x)$ and the second (singular) Legendre function $Q_n(x)$:

$$y(x) = C_1 P_n(x) + C_2 Q_n(x).$$

For $n = 1$,

$$P_1(x) = x,$$

and a (standard) closed form for the second independent solution $Q_1(x)$ is

$$Q_1(x) = \frac{1}{2}x \ln \frac{1+x}{1-x} - 1.$$

(One can obtain Q_1 by reduction of order starting from $y_1 = x$; the integral representation leads to the logarithmic form above.)

Thus the general solution is

$$y(x) = C_1 x + C_2 \left(\frac{1}{2}x \ln \frac{1+x}{1-x} - 1 \right), \quad |x| < 1.$$



Final Examination – Sample Paper

Total Marks : 60

Time Allotted : 3 hours

Instructions:

- The question paper consists of 3 sections. It is compulsory for students to attempt all questions of Section A and Section B.
- Section A has 5 questions of 2 marks each, Section B has 4 questions of 5 marks each, and Section C has 4 questions of 10 marks each (out of which any 3 are to be attempted).
- Question no. 10 & 11 of Section C are compulsory to be attempted.
- Students to attempt ANY ONE question from question no. 12 & question no. 13 of Section C.

Section A (5x2 = 10 marks)

1. Find the rank of the matrix:

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

2. For $T(x, y, z) = (x + y, y + z)$, find kernel and range.
3. Define complementary function and particular integral.
4. Check if $(1 + y^2)dx - (x + y^2)dy = 0$ is exact.
5. Find the general solution of $y' - y \tan x = \sin x$.

Section B (4x5 = 20 marks)

6. Solve the system using matrix inversion:

$$\begin{cases} 2x + 3y = 8 \\ x + 4y = 7 \end{cases}$$



7. If $T_1(x, y) = (x + y, x - y)$ and $T_2(u, v) = (2u, 3v)$, find $T_2(T_1(x, y))$.
8. Reduce $(2xy + y^2)dx + (x^2 + xy)dy = 0$ to exact form using an integrating factor.
9. Solve using variation of parameters: $y'' + y = \tan x$.

Section C (3x10 = 30 marks)

10. Obtain the echelon form and find the rank of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 0 & 5 \\ 3 & 6 & -1 & 8 \end{bmatrix}$$

11. Find the Eigenvalues and Eigenvectors of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

and hence diagonalize it.

12. Solve $\frac{dy}{dx} + y = e^{2x}y^2$ (Bernoulli form).

OR

13. Prove that if y_1 and y_2 are two solutions of a homogeneous linear DE, then $c_1y_1 + c_2y_2$ is also a solution.