

Eigen Values and Eigen Vectors

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1. Characteristic Equation $\rightarrow$

If A is a square matrix of order n , we can form the matrix $A - \lambda I$, where λ is a scalar and I is the unit matrix of order n . The determinant of this matrix equated to zero, i.e.

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix} = 0$$

is called the characteristic equation of A .

On expanding the determinant, the characteristic equation can be written as a polynomial equation of degree n in λ of the form

$$(-1)^n \lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + \dots + k_n = 0.$$

The roots of this equation are called the characteristic roots or latent roots or eigen values of A .

2. Eigen Vectors $\rightarrow$

Consider the linear transformation $B = AX$ — (1) which transforms the column vector X into the column vector B . In practice, we are often required to find those vectors X which transform into scalar multiples of themselves.

Let X be such a vector which transforms into λX (λ being a non-zero scalar) by the transformation ①. Then

$$BX = \lambda X \quad \text{--- ②}$$

From ① and ②,

$$AX = \lambda X \Rightarrow AX - \lambda IX = 0$$

$$\Rightarrow (A - \lambda I)X = 0 \quad \text{--- ③}$$

This matrix equation gives n homogeneous linear equations

$$\left. \begin{aligned} (a_{11} - \lambda)x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 + \dots + a_{2n}x_n &= 0 \\ \vdots &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + (a_{nn} - \lambda)x_n &= 0 \end{aligned} \right\} \quad \text{--- ④}$$

These equations will have a non-trivial solution only if the coefficient matrix $A - \lambda I$ is singular. i.e. if

$$|A - \lambda I| = 0 \quad \text{--- ⑤}$$

This is the characteristic equation of the matrix A and has n roots which are the eigen-values of A . Corresponding to each root of

⑤, the homogeneous system ③ has a non-zero solution.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ which is called an eigen vector or latent vector.}$$

Note: $\rightarrow$ If X is a solution of ③, then so is KX , where K is an arbitrary constant. Thus, the eigen vectors corresponding to an eigen-value is not unique.

Ex: Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

[U.P.T.U., 2001 (B.P)]

Solⁿ: The characteristic equation of the given matrix A is $|A - \lambda I| = 0$

$$\text{or } \begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

$$\text{or, } (3-\lambda)(2-\lambda)(5-\lambda) = 0$$

The eigen values is
 $\lambda = 2, 3, 5$

Now the equation

$[A - \lambda I]X = 0$, for the matrix A is

$$\begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- ①}$$

Put $\lambda = 2$ in equation ①, we get-

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \rightarrow R_3 - \frac{1}{2}R_2$$

$$\Rightarrow x + y + 4z = 0, \quad 6z = 0 \Rightarrow z = 0$$

$$\Rightarrow x + y = 0 \quad \text{or} \quad x = -y, \quad \text{taking } y = -1$$

The eigen vector corresponding to $\lambda = 2$, is given by

$$X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Put $\lambda = 3$ in (1), we get-

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow y + 4z = 0, \quad -y + 6z = 0, \quad 2z = 0 \quad \text{or} \quad z = 0$$

$$\Rightarrow \text{taking } x = 1, \quad y = 0, \quad z = 0$$

The eigen vector corresponding to $\lambda = 3$, is given by

$$X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Put $\lambda = 5$ in (1), we get-

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2x + y + 4z = 0, \quad -3y + 6z = 0$$

Taking $z = k$, then solving, we get- $x = 3k, \quad y = 2k, \quad z = k$

The eigen vector corresponding to $\lambda = 5$ is given by

$$X_3 = \begin{bmatrix} 3k \\ 2k \\ k \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Q: Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Solⁿ: The characteristic equation of the given matrix A is

$$|A - \lambda I| = 0$$

$$\text{or } \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$$

$$\text{or } (-2-\lambda) [-\lambda(1-\lambda) - 12] - 2 [-2\lambda - 6] - 3 [-4 + 1(1-\lambda)] = 0$$

$$\text{or, } \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

By trial, $\lambda = -3$ satisfies it.

$$\therefore (\lambda + 3)(\lambda^2 - 2\lambda - 15) = 0 \Rightarrow (\lambda + 3)(\lambda + 3)(\lambda - 5) = 0$$

$$\Rightarrow \lambda = -3, -3, 5$$

Thus, the eigen values of A are $-3, -3, 5$.

Corresponding to $\lambda = -3$, the eigen vectors are given by

$$[A + 3I] X = 0$$

$$\text{or, } \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

we get- only one independent equation

$$x + 2y - 3z = 0$$

choosing $y = 0$, we have $x - 3z = 0$

$$\therefore \frac{x}{3} = \frac{y}{0} = \frac{z}{1} \text{ giving the eigen vector } (3, 0, 1)$$

choosing $z = 0$, we have $x + 2y = 0$

$$\therefore \frac{x}{2} = \frac{y}{-1} = \frac{z}{0} \text{ giving the eigen vector } (2, -1, 0)$$

Any other eigen vector corresponding to $\lambda = -3$ will be a linear combination of these two.

Corresponding to $\lambda = 5$, the eigen vectors are given by

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} -7x + 2y - 3z &= 0 \\ 2x - 4y - 6z &= 0 \\ -x - 2y - 5z &= 0 \end{aligned}$$

Since only two of them are independent, we can omit one of them.

From first two equations, we have

$$\frac{x}{-12-12} = \frac{y}{-6-42} = \frac{z}{28-4} \quad \text{or} \quad \frac{x}{-1} = \frac{y}{-2} = \frac{z}{1}$$

$$\text{or } \frac{x}{1} = \frac{y}{2} = \frac{z}{-1} \text{ giving the eigen vector } (1, 2, -1).$$

$\lambda_1 = 0, \lambda_2 = 0$. Hence, the given vectors are linearly independent.

5. CAYLEY-HAMILTON THEOREM : $\rightarrow$

Every square matrix satisfies its characteristic equation.

If $|A - \lambda I| = (-1)^n [\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots + a_n]$ be the characteristic polynomial of an $n \times n$ matrix $A = [a_{ij}]$, then the matrix equation $x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n I = 0$ is satisfied by $x = A$.

i.e. $A^n + a_1 A^{n-1} + a_2 A^{n-2} + \dots + a_n I = 0$

Ex: Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

and hence find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

Solⁿ → The characteristic equation of A is $|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\text{or, } (2-\lambda) [(1-\lambda)(2-\lambda)] - 1(0) + 1[0 - 1 + \lambda] = 0$$

$$\text{or, } \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley-Hamilton theorem,

$$A^3 - 5A^2 + 7A - 3I = 0 \quad \text{--- (1)}$$

$$\text{Now, } A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$

$$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + (A^2 + A + I)$$

$$= A^5(0) + A(0) + (A^2 + A + I) \quad [\text{Using equation (1)}]$$

$$= A^2 + A + I$$

$$\Rightarrow A^2 + A + I = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5+2+1 & 4+1+0 & 4+1+0 \\ 0+0+0 & 1+1+1 & 0+0+0 \\ 4+1+0 & 4+1+0 & 5+2+1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

— Ans

Ex: Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Hence find A^{-1} .

Solⁿ → The characteristic equation of A is $|A - \lambda I| = 0$

ie
$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

On simplification we get,

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

To verify Cayley-Hamilton theorem, we have to show that—

$$A^3 - 6A^2 + 9A - 4I = 0$$

————— ①

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 22 & -22 & -21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 22 & -22 & -21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6 \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$+ 9 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -22 & -21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

This verifies Cayley-Hamilton theorem.

Now, multiplying both sides of ① by A^{-1} , we have

$$A^2 - 6A + 9I - 4A^{-1} = 0$$

$$\Rightarrow 4A^{-1} = A^2 - 6A + 9I$$

$$4A^{-1} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow 4A^{-1} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \quad \underline{\text{Ans}}$$

Ex: Use Cayley-Hamilton theorem to find out the inverse of the following matrix

$$A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

[U.P.T.V. 2002]

Solⁿ → The characteristic equation of the given matrix A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 4-\lambda & 3 & 1 \\ 2 & 1-\lambda & -2 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

or,

$$(4-\lambda)\{(1-\lambda)^2+4\}-3\{2-2\lambda+2\}+1\{4-1+\lambda\}=0$$

on simplification we get-

$$\lambda^3-6\lambda^2+6\lambda-11=0 \quad \text{--- (1)}$$

By Cayley-Hamilton theorem,

$$A^3-6A^2+6A-11I=0$$

$$\text{or } 11I = A^3-6A^2+6A$$

Multiplying both sides by A^{-1} , we get-

$$11A^{-1} = A^2-6A+6I \quad \text{--- (2)}$$

$$A^2 = A \times A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix}$$

From equation (2), we get-

$$11A^{-1} = \begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix} - 6 \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} + 6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{or, } 11A^{-1} = \begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{11} \begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{bmatrix}$$

Ans

Ex: Verify Cayley-Hamilton's theorem for the following matrix A and compute $2A^8 - 3A^5 + A^4 + A^2 - 4I$; where

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Solⁿ: → The characteristic equation of the given matrix A is

$$|A - \lambda I| = 0$$

or
$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{vmatrix} = 0$$

or,
$$(1-\lambda) \{ (-1-\lambda)(-\lambda) - 1 \} = 0$$

$$\Rightarrow \lambda^3 - 2\lambda + 1 = 0 \quad \text{--- (1)}$$

By Cayley-Hamilton theorem

$$A^3 - 2A + I = 0 \quad \text{--- (2)}$$

$$\therefore A^2 = A \cdot A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & -3 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$

Now,

$$A^3 - 2A + I = \begin{bmatrix} 1 & 0 & 4 \\ 0 & -3 & 2 \\ 0 & 2 & -1 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 4 \\ 0 & -3 & 2 \\ 0 & 2 & -1 \end{bmatrix} + \begin{bmatrix} -2 & 0 & -4 \\ 0 & 2 & -2 \\ 0 & -2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A^3 - 2A + I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

where O is the null matrix. Hence, the Cayley-Hamilton theorem is verified.

From equation (2), we have

$$A^3 = 2A - I \quad \text{--- (3)}$$

Multiplying both sides by (3) by A, A^2, A^3, A^4, A^5 respectively,

we have,

$$A^4 = 2A^2 - A$$

$$A^5 = 2A^3 - A^2$$

$$A^6 = 2A^4 - A^3$$

$$A^7 = 2A^5 - A^4$$

$$A^8 = 2A^6 - A^5$$

&

Now,

$$\begin{aligned} 2A^8 - 3A^5 + A^4 + A^2 - 4I &= 2(2A^6 - A^5) - 3A^5 + A^4 + A^2 - 4I \\ &= 4A^6 - 5A^5 + A^4 + A^2 - 4I \\ &= 4(2A^4 - A^3) - 5(2A^3 - A^2) + A^4 + A^2 - 4I \\ &= 9A^4 - 14A^3 + 6A^2 - 4I \\ &= 9(2A^2 - A) - 14(2A - I) + 6A^2 - 4I \\ &= 24A^2 - 37A + 10I \end{aligned}$$

$$\Rightarrow 24A^2 - 37A + 10I = 24 \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - 37 \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} + 10 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 48 & -26 \\ 0 & 95 & -61 \\ 0 & -61 & 34 \end{bmatrix}$$

Ans

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

3. Linearly Dependent of Vectors :-

The set of vectors (matrices) $x_1, x_2, x_3, \dots, x_n$ are said to be dependent if

- (i) all the vectors (row or column matrices) are of the same order.
- (ii) n scalar's $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ (not all zero) exists such that

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \dots + \lambda_n x_n = 0$$

where 0 is the zero vector.

4. Linearly Independent of Vectors :-

The set of vectors $x_1, x_2, x_3, \dots, x_n$ are said to be linearly independent if and only if there exists n scalars $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ (all zero) exist such that

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \dots + \lambda_n x_n = 0$$

Ex : Show that the vectors, $x_1 = (1, 2, 4)$, $x_2 = (2, -1, 3)$, $x_3 = (0, 1, 2)$ and $x_4 = (-3, 7, 2)$ are linearly dependent. Find the relation between them. [U.P.T.U. 2003]

Solⁿ : Consider the matrix equation

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 = 0 \quad \text{--- (1)}$$

$$\Rightarrow \lambda_1 (1, 2, 4) + \lambda_2 (2, -1, 3) + \lambda_3 (0, 1, 2) + \lambda_4 (-3, 7, 2) = (0, 0, 0)$$

Equating the corresponding components we have

$$\begin{aligned}\lambda_1 + 2\lambda_2 + 0\lambda_3 - 3\lambda_4 &= 0 \\ 2\lambda_1 - \lambda_2 + \lambda_3 + 7\lambda_4 &= 0 \\ 4\lambda_1 + 3\lambda_2 + 2\lambda_3 + 2\lambda_4 &= 0\end{aligned}$$

This is the homogeneous system $A\lambda = 0$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{aligned} R_2 &\rightarrow R_2 - 2R_1 \\ R_3 &\rightarrow R_3 - 4R_1 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \rightarrow R_3 - R_2$$

$$\lambda_1 + 2\lambda_2 + 0\lambda_3 - 3\lambda_4 = 0$$

$$-5\lambda_2 + \lambda_3 + 13\lambda_4 = 0$$

$$\lambda_3 + \lambda_4 = 0$$

$$\text{Let } \lambda_4 = P, \quad \lambda_3 + P = 0 \Rightarrow \lambda_3 = -P$$

$$-5\lambda_2 - P + 13P = 0 \Rightarrow \lambda_2 = \frac{12P}{5}$$

$$\lambda_1 + \frac{24P}{5} - 3P = 0 \Rightarrow \lambda_1 = -\frac{9P}{5}$$

Hence, the given vectors are linearly dependent.

Putting these values in equation ①, we get-

$$-\frac{9}{5}P x_1 + \frac{12}{5}P x_2 - Px_3 + Px_4 = 0$$

$$\text{or, } -9x_1 + 12x_2 - 5x_3 + 5x_4 = 0$$

$$\text{or, } 9x_1 - 12x_2 + 5x_3 - 5x_4 = 0$$

Ans.

Ex → Show that the vectors, $x_1 = [1, -1, 2, 0]$, $x_2 = [2, 1, 1, 1]$, $x_3 = [3, -1, 2, -1]$, $x_4 = [3, 0, 3, 1]$ are linearly dependent- and find the relation.

Solⁿ: Consider the matrix equation

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 = 0 \quad \text{————— ①}$$

$$\Rightarrow \lambda_1(1, -1, 2, 0) + \lambda_2(2, 1, 1, 1) + \lambda_3(3, -1, 2, -1) + \lambda_4(3, 0, 3, 1) = (0, 0, 0, 0)$$

Equating the corresponding components, we have

$$\lambda_1 + 2\lambda_2 + 3\lambda_3 + 3\lambda_4 = 0$$

$$-\lambda_1 + \lambda_2 - \lambda_3 + 0\lambda_4 = 0$$

$$2\lambda_1 + \lambda_2 + 2\lambda_3 + 3\lambda_4 = 0$$

$$0\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4 = 0$$

This is the Homogeneous system $A\lambda = 0$

$$\begin{bmatrix} 1 & 2 & 3 & 3 \\ -1 & 1 & -1 & 0 \\ 2 & 1 & 2 & 3 \\ 0 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 3 & 2 & 3 \\ 0 & -3 & -4 & -3 \\ 0 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{array}{l} R_2 \rightarrow R_2 + R_1 \\ R_3 \rightarrow R_3 - 2R_1 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & -3 & -4 & -3 \\ 0 & 3 & 2 & 3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad R_2 \leftrightarrow R_4$$

$$\begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{array}{l} R_3 \rightarrow R_3 + 3R_2 \\ R_4 \rightarrow R_4 - 3R_2 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \rightarrow (-1/7)R_3$$

$$\begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad R_4 \rightarrow R_4 - 5R_3$$

$$\lambda_1 + 2\lambda_2 + 3\lambda_3 + 3\lambda_4 = 0$$

$$\lambda_2 - \lambda_3 + \lambda_4 = 0$$

$$\lambda_3 = 0$$

$$\lambda_4 = p, \quad \lambda_2 - 0 + p = 0 \Rightarrow \lambda_2 = -p$$

$$\lambda_1 - 2p + 0 + 3p = 0 \Rightarrow \lambda_1 = -p$$

Hence, the given vectors are linearly dependent.
Putting these values in equation (1), we get

$$-P X_1 - P X_2 + 0 X_3 + P X_4 = 0$$

$$\Rightarrow X_1 + X_2 - X_4 = 0$$

Ans

Ex: Is the system of vectors $X_1 = (1, -1, 1)$, $X_2 = (2, 1, 1)$, $X_3 = (3, 0, 2)$, linearly dependent?

Solⁿ: Consider the matrix equation

$$\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 = 0 \quad \text{--- (1)}$$

$$\Rightarrow \lambda_1 (1, -1, 1) + \lambda_2 (2, 1, 1) + \lambda_3 (3, 0, 2) = (0, 0, 0)$$

Equating the corresponding components, we have

$$\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0 \quad \text{--- (2)}$$

$$-\lambda_1 + \lambda_2 + 0\lambda_3 = 0 \quad \text{--- (3)}$$

$$\lambda_1 + \lambda_2 + 2\lambda_3 = 0 \quad \text{--- (4)}$$

The coefficient matrix A of these equations is

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 1(2-0) - 2(-2-0) + 3(-1-1)$$

$$|A| = 2 + 4 - 6 = 0$$

Therefore, the rank A is less than 3. Therefore, the given system of equations will possess a non-zero solution then from (3), we get-

$$-\lambda_1 + \lambda_2 = 0 \Rightarrow \lambda_1 = \lambda_2.$$

Taking $\lambda_2 = 1$, $\lambda_1 = 1$, $\lambda_3 = -1$.

Hence, the given vectors are linearly dependent.

Ans

Ex: Show that the vectors $x_1 = (3, 1, -4)$, $x_2 = (2, 2, -3)$, $x_3 = (0, -4, 1)$ are linearly dependent.

Solⁿ: Consider the matrix equation

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0 \quad \text{--- ①}$$

$$\Rightarrow \lambda_1 (3, 1, -4) + \lambda_2 (2, 2, -3) + \lambda_3 (0, -4, 1) = (0, 0, 0)$$

Equating the corresponding components, we have

$$3\lambda_1 + 2\lambda_2 + \lambda_3 \cdot 0 = 0$$

$$\lambda_1 + 2\lambda_2 - 4\lambda_3 = 0$$

$$-4\lambda_1 - 3\lambda_2 + \lambda_3 = 0$$

The coefficient matrix A of these equation is

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 2 & -4 \\ -4 & -3 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & 2 & 0 \\ 1 & 2 & -4 \\ -4 & -3 & 1 \end{vmatrix}$$

$$|A| = 3(2-12) - 2(1-16) + 0 \\ = -30 + 30 = 0$$

Therefore, the rank A is less than 3. Therefore the given system of equations will possess a non-zero solution then from (2),

$$3\lambda_1 + 2\lambda_2 = 0 \Rightarrow \lambda_1 = -\frac{2}{3}\lambda_2,$$

taking $\lambda_2 = 1$, $\lambda_1 = -\frac{2}{3}$

From (3), $-\frac{2}{3} + 2 - 4\lambda_3 = 0 \Rightarrow \lambda_3 = \frac{1}{3}$

Hence the given vectors are linearly dependent.

Ans.

Ex: Examine whether the following set of vectors is linearly independent.

(i) $(1, 2, 3, 4), (2, 0, 1, -2), (3, 2, 4, 2).$

(ii) $(1, 1, 0, 1), (1, 1, 1, 1), (-1, 1, 1, 1), (1, 0, 0, 1).$

(iii) $(2, 3, 6, -3, 4), (4, 2, 12, -3, 6), (4, 10, 12, -9, 10)$

Solⁿ: $\rightarrow$ Let each vectors represent a row of a matrix A . We reduce A to row echelon form. If all the rows of the row echelon form have some non-zero elements, then the given set of vectors are linearly independent.

(i) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & -2 \\ 3 & 2 & 4 & 2 \end{bmatrix} \approx \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -5 & -10 \\ 0 & -4 & -5 & -10 \end{bmatrix} \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array}$

$$\approx \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -5 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \rightarrow R_3 - R_2$$

Since all the rows in the row echelon form of A are not non-zero, the given set of vectors are linearly dependent. Ans

(ii)

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 + R_1 \\ R_4 \rightarrow R_4 - R_1 \end{array}$$

$$\approx \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_3 \approx \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 1 \end{bmatrix} \quad R_4 \rightarrow R_4 + \frac{1}{2}R_3$$

$$\approx \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_4 \rightarrow R_4 - \frac{1}{2}R_3$$

Since all the rows in the row echelon form of A are non-zero, the given set of vectors are linearly independent. Ans

(iii)

$$A = \begin{bmatrix} 2 & 3 & 6 & -3 & 4 \\ 4 & 2 & 12 & -3 & 6 \\ 4 & 10 & 12 & -9 & 10 \end{bmatrix} \approx \begin{bmatrix} 2 & 3 & 6 & -3 & 4 \\ 0 & -4 & 0 & 3 & -2 \\ 0 & 4 & 0 & -3 & 2 \end{bmatrix} \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 2R_1 \end{array}$$

$$\approx \begin{bmatrix} 2 & 3 & 6 & -3 & 4 \\ 0 & -4 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \rightarrow R_3 + R_4$$

Since all the rows in the row echelon form of A are not non-zero, the given set of vectors are linearly dependent.

Ans

← Exercise →

- ① Find all the solutions of the following systems of linear homogeneous equations $x+y+z=0$, $2x+5y+7z=0$, $2x-5y+3z=0$.
- ② Find all the solutions of the following systems of linearly homogeneous equations $x-2y+z-w=0$, $x+y-2z+3w=0$, $4x+y-5z+8w=0$, $5x-7y+2z-w=0$.
- ③ Solve the matrix method, the equations $3x+y+2z=3$, $2x-3y-z=-3$, $x+2y+z=4$.
- ④ Show that the equations $x-4y+7z=14$, $3x+8y-2z=13$, $7x-8y+26z=5$ are not consistent.
- ⑤ Test whether the following equations are consistent:
 $x+2y-z=3$, $3x-y+2z=1$, $2x-2y+3z=2$, $x-y+z=-1$
 If found consistent, find the solutions.
- ⑥ Apply the test of rank to show that the equations $x+y+z=8$, $x-y+2z=6$, $3x+5y-7z=14$ are consistent and solve them.



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