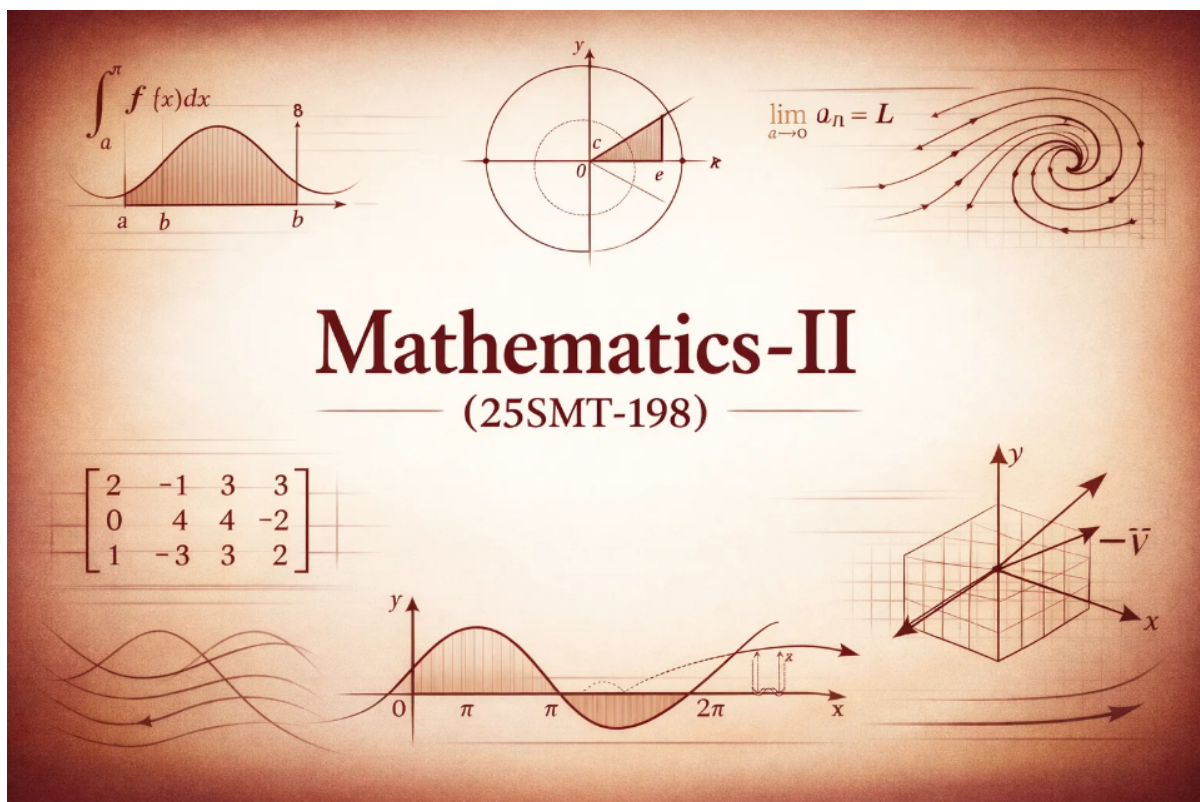




Unit 2 - Resources & Questions

Compiled by : Subhayu



Syllabus for Unit 2

Mathematics-II (25SMT-198)

Chapter 1 : Recursion and recurrence relations – Recurrence relations with constant coefficients, homogeneous and particular solutions, solutions of recurrence relation using generating functions.

Chapter 2 : Algebraic structures – Introduction to algebraic structures, semi group, monoid, group, congruence relations, subgroups, cosets, normal subgroups, cyclic groups, homomorphism, isomorphism and automorphism.

Self-Study Topic – Group Theory (Cayley's Theorem, Sylow's Theorems)

Chapter 1

Recursion and Recurrence Relations

1.1 Introduction to Recursion and Recurrence Relations

1.1.1 What is Recursion?

Recursion is a problem-solving technique where a problem is defined in terms of smaller instances of itself. It is widely used in computer science, especially in algorithms and data structures.

Example: The factorial of a number n is defined as:

$$n! = n \cdot (n - 1)!$$

with base case:

$$0! = 1$$

1.1.2 What is a Recurrence Relation?

A recurrence relation defines a sequence where each term is expressed as a function of its previous terms.

Example:

$$a_n = a_{n-1} + 2, \quad a_0 = 1$$

Key Idea

A recurrence relation + initial condition uniquely defines a sequence.

1.1.3 Why Study Recurrence Relations?

- To analyze recursive algorithms



- To model real-world processes (growth, decay, etc.)
- To understand time complexity in algorithms

1.2 Types of Recurrence Relations

1.2.1 Linear vs Non-linear Recurrence Relations

Linear Recurrence: A recurrence is linear if each term is a linear combination of previous terms.

$$a_n = 2a_{n-1} + 3a_{n-2}$$

Non-linear Recurrence:

$$a_n = a_{n-1}^2 + 1$$

1.2.2 Homogeneous vs Non-homogeneous Recurrence Relations

Homogeneous Recurrence: No extra function of n is present.

$$a_n - 3a_{n-1} + 2a_{n-2} = 0$$

Non-homogeneous Recurrence: Contains an additional term.

$$a_n - 3a_{n-1} + 2a_{n-2} = 5$$

Important Observation

Homogeneous $\Rightarrow$ RHS = 0

Non-homogeneous $\Rightarrow$ RHS $\neq 0$

1.3 Recurrence Relations with Constant Coefficients

1.3.1 Definition

A recurrence relation has constant coefficients if the coefficients of all terms are constants.

$$a_n + c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k} = f(n)$$

where $c_1, c_2, \dots, c_k$ are constants.



1.3.2 Example

$$a_n - 4a_{n-1} + 4a_{n-2} = 0$$

1.4 Homogeneous Recurrence Relations

1.4.1 General Form

$$a_n + c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k} = 0$$

1.4.2 Solution Method: Characteristic Equation

Step 1: Assume solution of form $a_n = r^n$

Step 2: Substitute into recurrence

Step 3: Form characteristic equation

$$r^k + c_1r^{k-1} + \cdots + c_k = 0$$

Step 4: Solve for roots

1.4.3 Case 1: Distinct Roots

If roots are $r_1, r_2, \dots, r_k$:

$$a_n = C_1r_1^n + C_2r_2^n + \cdots + C_kr_k^n$$

1.4.4 Case 2: Repeated Roots

If a root r is repeated m times:

$$a_n = (C_1 + C_2n + \cdots + C_mn^{m-1})r^n$$

1.4.5 Solved Example 1

Solve:

$$a_n - 5a_{n-1} + 6a_{n-2} = 0, \quad a_0 = 2, \quad a_1 = 5$$

Step 1: Characteristic Equation

$$r^2 - 5r + 6 = 0$$

Step 2: Solve

$$(r - 2)(r - 3) = 0 \Rightarrow r = 2, 3$$

**Step 3: General Solution**

$$a_n = C_1 2^n + C_2 3^n$$

Step 4: Apply Initial Conditions

For $n = 0$:

$$C_1 + C_2 = 2$$

For $n = 1$:

$$2C_1 + 3C_2 = 5$$

Solving:

$$C_1 = 1, \quad C_2 = 1$$

Final Answer:

$$a_n = 2^n + 3^n$$

Exam Tip

Always verify your solution using initial conditions — many mistakes happen here.

1.5 Non-Homogeneous Recurrence Relations

1.5.1 General Form

$$a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k} = f(n)$$

1.5.2 Solution Structure

$$a_n = a_n^{(h)} + a_n^{(p)}$$

where:

- $a_n^{(h)}$ = solution of homogeneous part
- $a_n^{(p)}$ = particular solution

1.6 Particular Solution Methods

1.6.1 Case 1: Constant RHS

If $f(n) = k$, assume:

$$a_n^{(p)} = A$$



1.6.2 Case 2: Polynomial RHS

If $f(n)$ is polynomial of degree m :

$$a_n^{(p)} = A_0 + A_1n + \cdots + A_mn^m$$

1.6.3 Case 3: Exponential RHS

If $f(n) = kr^n$:

$$a_n^{(p)} = Ar^n$$

1.6.4 Important Adjustment Rule

If assumed solution overlaps with homogeneous solution, multiply by n .

Common Mistake Alert

If r is already a root of the characteristic equation, multiply the particular solution by n (or higher power if repeated).

1.6.5 Solved Example 2

Solve:

$$a_n - 3a_{n-1} = 4$$

Step 1: Homogeneous Part

$$a_n - 3a_{n-1} = 0 \Rightarrow r - 3 = 0 \Rightarrow r = 3$$

$$a_n^{(h)} = C \cdot 3^n$$

Step 2: Particular Solution

Since RHS is constant, assume:

$$a_n^{(p)} = A$$

Substitute:

$$A - 3A = 4 \Rightarrow -2A = 4 \Rightarrow A = -2$$

Step 3: General Solution

$$a_n = C \cdot 3^n - 2$$



1.7 Final Thoughts and Intuition

1.7.1 How to Think About Recurrences

- Homogeneous part = natural behavior of system
- Particular part = external forcing (extra input)

Conceptual Insight

Think of recurrence relations like differential equations — homogeneous solution gives the system's natural response, while the particular solution adjusts it to match external conditions.

1.7.2 Quick Summary

- Form characteristic equation
- Solve roots
- Write homogeneous solution
- Guess particular solution
- Combine and apply initial conditions

1.8 Solutions of Recurrence Relations using Generating Functions

1.8.1 Introduction to Generating Functions

Generating functions provide a powerful and elegant method to solve recurrence relations, especially when traditional methods become complicated.

Definition: For a sequence $\{a_n\}$, the generating function $G(x)$ is defined as:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

Key Idea

A generating function encodes an entire sequence into a single algebraic expression.



1.8.2 Why Use Generating Functions?

- Converts recurrence relations into algebraic equations
- Easier manipulation using algebraic techniques
- Useful for combinatorics and discrete mathematics

1.9 Method of Solving Recurrence using Generating Functions

1.9.1 General Steps

1. Define generating function:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

2. Multiply recurrence relation by x^n
3. Sum over all n
4. Express everything in terms of $G(x)$
5. Solve for $G(x)$
6. Expand $G(x)$ to find a_n

Important Insight

The core trick is to shift indices cleverly so that sums align with $G(x)$.

1.10 Basic Manipulations of Generating Functions

1.10.1 Shifting Property

$$\sum_{n=1}^{\infty} a_{n-1} x^n = xG(x)$$

1.10.2 Example Transformation

If:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

Then:

$$\sum_{n=2}^{\infty} a_{n-2} x^n = x^2 G(x)$$



1.11 Solved Example 1

Solve:

$$a_n = a_{n-1} + 1, \quad a_0 = 0$$

1.11.1 Step 1: Define Generating Function

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

1.11.2 Step 2: Multiply by x^n and Sum

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} x^n$$

1.11.3 Step 3: Convert to $G(x)$

LHS:

$$G(x) - a_0 = G(x)$$

RHS:

$$xG(x) + \frac{x}{1-x}$$

1.11.4 Step 4: Solve Equation

$$G(x) = xG(x) + \frac{x}{1-x}$$

$$G(x)(1-x) = \frac{x}{1-x}$$

$$G(x) = \frac{x}{(1-x)^2}$$

1.11.5 Step 5: Expand

$$\frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} nx^n$$

Final Answer:

$$a_n = n$$

Quick Check

Always compare your final result with intuition — here, repeatedly adding 1 clearly gives $a_n = n$.



1.12 Solved Example 2

Solve:

$$a_n = 2a_{n-1}, \quad a_0 = 1$$

1.12.1 Step 1: Generating Function

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

1.12.2 Step 2: Apply Recurrence

$$\sum_{n=1}^{\infty} a_n x^n = 2 \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$G(x) - 1 = 2xG(x)$$

1.12.3 Step 3: Solve

$$G(x)(1 - 2x) = 1$$

$$G(x) = \frac{1}{1 - 2x}$$

1.12.4 Step 4: Expand

$$\frac{1}{1 - 2x} = \sum_{n=0}^{\infty} 2^n x^n$$

Final Answer:

$$a_n = 2^n$$

1.13 Generating Functions for Non-Homogeneous Recurrence

1.13.1 Example with Constant Term

Solve:

$$a_n = a_{n-1} + n, \quad a_0 = 0$$

1.13.2 Idea

Split into:



- Known generating function for n
- Combine algebraically

Concept Booster

You should memorize common generating functions:

$$\sum x^n = \frac{1}{1-x}, \quad \sum nx^n = \frac{x}{(1-x)^2}$$

1.14 Common Generating Functions

1.14.1 Standard Forms

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}$$

$$\sum_{n=0}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}$$

1.14.2 Why These Matter

These help in quickly identifying patterns and extracting coefficients.

1.15 Advantages and Limitations

1.15.1 Advantages

- Systematic and structured approach
- Works well for complex recurrences
- Useful in combinatorics

1.15.2 Limitations

- Algebra can become complicated
- Requires familiarity with series expansion



1.16 Final Intuition

Deep Insight

Generating functions transform sequences into algebra — turning a recursive problem into an algebraic one.

1.16.1 Quick Summary

- Define generating function
- Convert recurrence into equation
- Solve algebraically
- Expand to get a_n

Chapter 2

Algebraic Structures

2.1 Introduction to Algebraic Structures

2.1.1 What is an Algebraic Structure?

An algebraic structure consists of a non-empty set along with one or more operations defined on it that satisfy certain properties.

Definition: An algebraic structure is a pair $(S, *)$ where:

- S is a non-empty set
- $*$ is a binary operation on S

Binary Operation: A binary operation $*$ on a set S is a function:

$$* : S \times S \rightarrow S$$

Example:

- $(\mathbb{Z}, +) \rightarrow$ Integers under addition
- $(\mathbb{N}, \times) \rightarrow$ Natural numbers under multiplication

Core Idea

Closure is the first requirement of any algebraic structure: applying the operation on elements of the set must produce an element within the same set.

2.1.2 Important Properties of Operations

- **Closure:** $a * b \in S$
- **Associativity:** $(a * b) * c = a * (b * c)$



- **Commutativity:** $a * b = b * a$
- **Identity Element:** $a * e = e * a = a$
- **Inverse Element:** $a * a^{-1} = e$

2.2 Semigroup

2.2.1 Definition

A semigroup is an algebraic structure $(S, *)$ such that:

- S is non-empty
- $*$ is associative

$$(a * b) * c = a * (b * c) \quad \forall a, b, c \in S$$

2.2.2 Examples

- $(\mathbb{Z}, +)$ is a semigroup
- $(\mathbb{N}, \times)$ is a semigroup

2.2.3 Non-Example

$$a * b = a - b \quad \text{on } \mathbb{Z}$$

Not associative, hence not a semigroup.

Key Insight

Associativity is the ONLY requirement for a semigroup — identity is not required.

2.3 Monoid

2.3.1 Definition

A monoid is a semigroup with an identity element.

$(S, *)$ is a monoid if:

- Associativity holds
- There exists $e \in S$ such that $a * e = e * a = a$



2.3.2 Examples

- $(\mathbb{Z}, +)$ with identity 0
- $(\mathbb{N}, \times)$ with identity 1

2.3.3 Solved Example

Check whether $(\mathbb{N}, +)$ is a monoid.

Solution:

- Closure: Yes
- Associativity: Yes
- Identity: $0 \notin \mathbb{N}$ (if $\mathbb{N} = \{1, 2, \dots\}$)

Conclusion: Not a monoid (but a semigroup)

Exam Insight

Always check whether 0 is included in $\mathbb{N}$ — definitions vary!

2.4 Group

2.4.1 Definition

A group is a monoid where every element has an inverse.

$(S, *)$ is a group if:

- Closure
- Associativity
- Identity exists
- Inverse exists for every element

2.4.2 Abelian Group

If additionally:

$$a * b = b * a$$

then the group is called **Abelian (commutative)**.



2.4.3 Examples

- $(\mathbb{Z}, +) \rightarrow$ Abelian group
- $(\mathbb{R} \setminus \{0\}, \times) \rightarrow$ Abelian group

2.4.4 Non-Abelian Example

Matrix multiplication:

$$AB \neq BA$$

2.4.5 Solved Example

Check whether $(\mathbb{Z}, \times)$ is a group.

Solution:

- Closure: Yes
- Associativity: Yes
- Identity: 1 exists
- Inverse: Not all integers have multiplicative inverse

Conclusion: Not a group

Common Trap

Do NOT assume existence of inverse — always verify it for every element.

2.5 Congruence Relations

2.5.1 Definition

A congruence relation is an equivalence relation that is compatible with a given operation.

For integers:

$$a \equiv b \pmod{n} \iff n \mid (a - b)$$

2.5.2 Properties

Congruence relation satisfies:

- Reflexive: $a \equiv a$
- Symmetric: $a \equiv b \Rightarrow b \equiv a$
- Transitive: $a \equiv b, b \equiv c \Rightarrow a \equiv c$



2.5.3 Compatibility with Operation

If:

$$a \equiv b \pmod{n}, \quad c \equiv d \pmod{n}$$

Then:

$$a + c \equiv b + d \pmod{n}$$

$$a \cdot c \equiv b \cdot d \pmod{n}$$

2.5.4 Solved Example

Check whether:

$$17 \equiv 5 \pmod{6}$$

Solution:

$$17 - 5 = 12$$

$$6 \mid 12 \Rightarrow \text{True}$$

Conclusion: $17 \equiv 5 \pmod{6}$

Conceptual Insight

Congruence relations group integers into equivalence classes — this is the foundation of modular arithmetic.

2.6 Hierarchy of Algebraic Structures

2.6.1 Understanding the Flow

$$\text{Semigroup} \subset \text{Monoid} \subset \text{Group}$$

- Semigroup $\rightarrow$ only associativity
- Monoid $\rightarrow$ associativity + identity
- Group $\rightarrow$ associativity + identity + inverse

Final Takeaway

Every group is a monoid, every monoid is a semigroup — but not vice versa.

2.7 Quick Summary

- Algebraic structure = set + operation



- Semigroup = associative
- Monoid = associative + identity
- Group = associative + identity + inverse
- Congruence relation = equivalence relation compatible with operation

2.8 Subgroups

2.8.1 Definition

A subset H of a group $(G, *)$ is called a **subgroup** if H itself forms a group under the same operation.

2.8.2 Subgroup Test

A non-empty subset $H \subseteq G$ is a subgroup if:

- $a, b \in H \Rightarrow a * b \in H$ (Closure)
- $a \in H \Rightarrow a^{-1} \in H$ (Inverse)

Shortcut Test:

$$a, b \in H \Rightarrow a * b^{-1} \in H$$

2.8.3 Example

Let $G = (\mathbb{Z}, +)$ and $H = 2\mathbb{Z}$ (even integers).

$$a, b \in H \Rightarrow a - b \in H$$

Hence, H is a subgroup.

Quick Trick

In additive groups, the subgroup test becomes: $a - b \in H$.

2.9 Cosets

2.9.1 Definition

Let H be a subgroup of G .



Left Coset:

$$aH = \{a * h \mid h \in H\}$$

Right Coset:

$$Ha = \{h * a \mid h \in H\}$$

2.9.2 Example

Let $G = \mathbb{Z}$, $H = 3\mathbb{Z}$.

$$1 + H = \{\dots, -5, -2, 1, 4, 7, \dots\}$$

2.9.3 Properties of Cosets

- All cosets have same size
- Cosets partition the group
- Either two cosets are identical or disjoint

Important Result

Number of distinct cosets = $\frac{|G|}{|H|}$ (Lagrange's idea for finite groups)

2.10 Normal Subgroups

2.10.1 Definition

A subgroup N of G is called **normal** if:

$$aN = Na \quad \forall a \in G$$

2.10.2 Equivalent Condition

$$aN a^{-1} = N$$

2.10.3 Example

In $(\mathbb{Z}, +)$, every subgroup is normal because:

$$a + H = H + a$$



2.10.4 Importance

Normal subgroups allow construction of quotient groups.

Key Insight

All subgroups of Abelian groups are normal.

2.11 Cyclic Groups

2.11.1 Definition

A group G is cyclic if there exists an element $g \in G$ such that:

$$G = \langle g \rangle = \{g^n \mid n \in \mathbb{Z}\}$$

2.11.2 Generator

The element g is called a **generator** of the group.

2.11.3 Examples

- $(\mathbb{Z}, +)$ generated by 1
- $\mathbb{Z}_n$ generated by 1

2.11.4 Properties

- Every cyclic group is Abelian
- Finite cyclic group of order n has $\phi(n)$ generators

Concept Insight

Cyclic groups are the simplest groups — generated by just one element.

2.12 Homomorphism

2.12.1 Definition

A function $\phi : G \rightarrow G'$ is a homomorphism if:

$$\phi(a * b) = \phi(a) * \phi(b)$$



2.12.2 Example

$$\phi : \mathbb{Z} \rightarrow \mathbb{Z}_n, \quad \phi(a) = a \bmod n$$

2.12.3 Kernel and Image

- Kernel:

$$\ker(\phi) = \{g \in G \mid \phi(g) = e'\}$$

- Image:

$$\text{Im}(\phi) = \{\phi(g) \mid g \in G\}$$

Important Result

Kernel of a homomorphism is always a normal subgroup.

2.13 Isomorphism

2.13.1 Definition

A bijective homomorphism $\phi : G \rightarrow G'$ is called an **isomorphism**.

$$G \cong G'$$

2.13.2 Meaning

Two groups are structurally the same (only elements differ in naming).

2.13.3 Example

$$(\mathbb{Z}, +) \cong (2\mathbb{Z}, +)$$

2.13.4 Properties Preserved

- Identity
- Inverses
- Operation structure

Key Idea

Isomorphic groups are essentially identical in behavior.



2.14 Automorphism

2.14.1 Definition

An automorphism is an isomorphism from a group to itself.

$$\phi : G \rightarrow G$$

2.14.2 Example

For $(\mathbb{Z}, +)$:

$$\phi(x) = -x$$

2.14.3 Properties

- Bijective
- Structure-preserving

Insight

Automorphisms describe symmetries within a group.

2.15 Summary and Connections

2.15.1 Big Picture

- Subgroups $\rightarrow$ smaller groups inside a group
- Cosets $\rightarrow$ partitions of group
- Normal subgroups $\rightarrow$ enable quotient groups
- Cyclic groups $\rightarrow$ simplest structure
- Homomorphism $\rightarrow$ structure-preserving maps
- Isomorphism $\rightarrow$ structural equality
- Automorphism $\rightarrow$ internal symmetry

Final Takeaway

Group theory studies structure — not just elements. These concepts help us understand how groups relate, transform, and simplify.

Self-Study Topic: Group Theory (Cayley's Theorem, Sylow's Theorems)

Introduction

Why These Theorems Matter

Cayley's Theorem and Sylow's Theorems are fundamental results in group theory that help us understand the **structure** of finite groups.

- Cayley's Theorem connects abstract groups with permutation groups
- Sylow's Theorems give deep insight into subgroup structure

Big Picture

These theorems answer: *"How are groups built and how can we analyze their internal structure?"*

Cayley's Theorem

Statement of Cayley's Theorem

Every finite group G of order n is isomorphic to a subgroup of the symmetric group S_n .

Understanding the Statement

- S_n = group of all permutations on n elements
- Cayley says: every group can be represented as permutations



Idea of Proof (Intuition)

For each element $g \in G$, define a function:

$$\phi_g(x) = g * x$$

This function rearranges elements of $G \rightarrow$ hence a permutation.

- Each g gives a permutation
- Mapping $g \rightarrow \phi_g$ preserves operation

Key Consequence

$$G \cong \text{a subgroup of } S_n$$

Concept Insight

Abstract groups are not “mysterious” — they can always be viewed as permutation groups.

Solved Example

Let $G = \mathbb{Z}_3 = \{0, 1, 2\}$ under addition mod 3.

Define:

$$\phi_1(x) = 1 + x \pmod{3}$$

Then:

$$\phi_1 : (0 \rightarrow 1, 1 \rightarrow 2, 2 \rightarrow 0)$$

This is a permutation in S_3 .

Hence, $\mathbb{Z}_3$ is isomorphic to a subgroup of S_3 .

Symmetric Groups

Definition

The symmetric group S_n is the group of all permutations of n elements.

$$|S_n| = n!$$



Example

$$S_3 = \{\text{all permutations of } \{1, 2, 3\}\}$$

Important Note

Cayley's theorem uses S_n as a “universal container” for all groups of order n .

Sylow's Theorems

Motivation

Sylow's theorems help determine:

- Existence of subgroups of certain orders
- Number of such subgroups

2.15.2 Preliminaries

Let:

$$|G| = p^k m$$

where:

- $p = \text{prime}$
- $p \nmid m$

First Sylow Theorem (Existence)

Statement

If $|G| = p^k m$, then G has a subgroup of order p^k .

Meaning

There exists at least one subgroup whose size is the highest power of p dividing $|G|$.

Key Idea

Every finite group contains “maximal p -subgroups” (called Sylow p -subgroups).



Second Sylow Theorem (Conjugacy)

Statement

All Sylow p -subgroups are conjugate to each other.

$$\forall P, Q \text{ Sylow } p\text{-subgroups}, \exists g \in G \text{ such that } gPg^{-1} = Q$$

Meaning

All Sylow subgroups are structurally similar inside the group.

Insight

Sylow subgroups are not random — they are interconnected via group symmetry.

Third Sylow Theorem (Counting)

Statement

Let n_p be the number of Sylow p -subgroups. Then:

- $n_p \equiv 1 \pmod{p}$
- $n_p \mid m$

Interpretation

- Number of Sylow subgroups is restricted
- Helps in determining group structure

Exam Focus

Always check both conditions:

$$n_p \equiv 1 \pmod{p}, \quad n_p \mid m$$

Solved Example on Sylow Theorems

Problem

Let $|G| = 12 = 2^2 \cdot 3$. Find possible number of Sylow 3-subgroups.



2.15.3 Solution

Here:

$$p = 3, \quad k = 1, \quad m = 4$$

Using Sylow III:

$$n_3 \equiv 1 \pmod{3}, \quad n_3 \mid 4$$

Divisors of 4:

$$1, 2, 4$$

Check condition:

- $1 \equiv 1 \pmod{3}$
- $4 \equiv 1 \pmod{3}$

Final Answer:

$$n_3 = 1 \text{ or } 4$$

Interpretation

If $n_3 = 1$, then the Sylow subgroup is unique and hence normal.

Applications of Sylow Theorems

Why They Are Powerful

- Help classify finite groups
- Determine existence of normal subgroups
- Simplify complex group structures

Final Summary

Cayley vs Sylow

- Cayley $\rightarrow$ representation of groups as permutations
- Sylow $\rightarrow$ internal subgroup structure

Final Takeaway

Cayley shows how groups can be represented, while Sylow explains how groups are internally organized.



Quick Recap

- Cayley: Every group $\subseteq S_n$
- Sylow I: Existence of p -subgroups
- Sylow II: All are conjugate
- Sylow III: Number constraints

Notefity



Mid Semester Test 2 – Sample Paper

Total Marks : 20

Time Allotted : 1 hour

Instructions:

- The question paper consists of 2 sections: Section A and Section B.
- Section A has 5 questions of 2 marks each and Section B has 2 questions of 5 marks each.

Section A ($5 \times 2 = 10$ marks)

1. Define a recurrence relation and state its properties.
2. Show that the set of even integers forms a subgroup of $(\mathbb{Z}, +)$.
3. Find the general solution of the recurrence relation:

$$a_n - 4a_{n-1} + 4a_{n-2} = 0$$

and discuss the nature of its roots.

4. State the difference between groups, semigroups and subgroups with proper examples.
5. Find all automorphisms of the group $(\mathbb{Z}, +)$ and justify your answer.

Section B ($2 \times 5 = 10$ marks)

6. Solve the non-homogeneous recurrence relation:

$$a_n - 2a_{n-1} = 3, \quad a_0 = 2$$

7. Check whether $(\mathbb{N}, +)$ forms a monoid. Explain your answer carefully considering different definitions of $\mathbb{N}$.

Answer Key

Section A

1. Definition and Properties of Recurrence Relation

A recurrence relation is an equation that defines each term of a sequence in terms of its preceding terms.

Example:

$$a_n = a_{n-1} + 2$$

Properties:

- Requires initial condition(s) to uniquely determine the sequence
- Can be linear or non-linear
- Can be homogeneous (RHS = 0) or non-homogeneous (RHS $\neq 0$)

Key Point

Without initial conditions, a recurrence relation does not define a unique sequence.

2. Even Integers Form a Subgroup of $(\mathbb{Z}, +)$

Let $H = 2\mathbb{Z}$.

- **Closure:** If $a = 2m$, $b = 2n$, then $a + b = 2(m + n) \in H$
- **Identity:** $0 \in H$
- **Inverse:** If $a = 2m$, then $-a = -2m \in H$

Hence, H is a subgroup of $(\mathbb{Z}, +)$.

Shortcut

For additive groups, check: $a - b \in H$.



3. Solve Recurrence Relation

$$a_n - 4a_{n-1} + 4a_{n-2} = 0$$

Characteristic equation:

$$r^2 - 4r + 4 = 0 \Rightarrow (r - 2)^2 = 0$$

Nature of Roots: Repeated root $r = 2$

General Solution:

$$a_n = (C_1 + C_2n)2^n$$

Concept

Repeated roots introduce an extra factor of n in the solution.

4. Difference Between Semigroup, Group, and Subgroup

- **Semigroup:** Set with associative operation Example: $(\mathbb{N}, +)$
- **Group:** Semigroup + identity + inverse Example: $(\mathbb{Z}, +)$
- **Subgroup:** Subset of a group that itself forms a group Example: $2\mathbb{Z} \subset \mathbb{Z}$

Hierarchy

Semigroup $\subset$ Monoid $\subset$ Group

5. Automorphisms of $(\mathbb{Z}, +)$

Let $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$ be an automorphism.

Since $\mathbb{Z}$ is cyclic (generated by 1), ϕ is determined by $\phi(1)$.

Possible mappings:

$$\phi(1) = 1 \Rightarrow \phi(n) = n$$

$$\phi(1) = -1 \Rightarrow \phi(n) = -n$$

Thus, automorphisms are:

$$\phi(n) = n \quad \text{and} \quad \phi(n) = -n$$

Key Insight

Automorphisms of cyclic groups depend on images of generators.



Section B

6. Solve Non-Homogeneous Recurrence

$$a_n - 2a_{n-1} = 3, \quad a_0 = 2$$

Step 1: Homogeneous Solution

$$a_n - 2a_{n-1} = 0 \Rightarrow r - 2 = 0 \Rightarrow r = 2$$

$$a_n^{(h)} = C \cdot 2^n$$

Step 2: Particular Solution

Assume $a_n^{(p)} = A$

$$A - 2A = 3 \Rightarrow -A = 3 \Rightarrow A = -3$$

Step 3: General Solution

$$a_n = C \cdot 2^n - 3$$

Step 4: Apply Initial Condition

$$a_0 = C - 3 = 2 \Rightarrow C = 5$$

Final Answer:

$$a_n = 5 \cdot 2^n - 3$$

Check

Substitute back to verify correctness — always a good habit!

7. Is $(\mathbb{N}, +)$ a Monoid?

Case 1: $\mathbb{N} = \{1, 2, 3, \dots\}$

- Closure: Yes
- Associativity: Yes
- Identity: 0 not present

$\Rightarrow$ Not a monoid

Case 2: $\mathbb{N} = \{0, 1, 2, \dots\}$



- Closure: Yes
- Associativity: Yes
- Identity: 0 exists

⇒ It is a monoid

Exam Insight

Always clarify the definition of $\mathbb{N}$ before answering.

Notefity

Closing Note

I sincerely hope that these notes and the preparatory questions included alongside them prove helpful in strengthening your understanding and boosting your confidence for the examination you are preparing for.

As you revise, try not to rush through the content—pause at examples, work out the numericals on your own, and revisit sections that feel slightly uncomfortable. That discomfort usually signals learning in progress. Use these notes as a guide and a companion, not as something to be memorized line by line.

Wishing you the very best for your exams—may your preparation feel clear, structured, and calm when it matters most.

For future updates, refinements, and additional resources, keep checking **Noteify**:

<https://subhayu18.github.io/Noteify/>

If you have any doubts, suggestions, or simply wish to connect, feel free to reach out to me on LinkedIn:

<https://www.linkedin.com/in/subhayu-mukherjee-146421294/>

You can also use the direct contact options provided on the Noteify website itself.

Regards,
Subhayu